CONTRIBUTIONS TO MATHEMATICAL BIOSCIENCES

(Mathematical Bio-Economics, B-D-I-E Processes & Epidemics)

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CERTIFICATE

This is to certify that the work presented in this thesis entitled 'CONTRIBUTIONS TO MATHEMATICAL BIOSCIENCES' by Mr. Mohd. Saleem has been carried out under my supervision and that this has not been submitted elsewhere for a degree or diploma.

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CHAPTER I

INTRODUCTION

1.1 FOUR AREAS OF MATHEMATICAL BIOSCIENCES CONSIDERED IN THE PRESENT THESIS

In the present thesis, we have made contribution to the following four areas of Mathematical Biosciences, namely,

(i) optimal exploitation of renewable resources like fisheries

(ii) optimal exploitation of exhaustible resources like mines

(iii) linear birth-death-immigration-emigration processes

(iv) spread of epidemics.

In each of these areas, we have investigated the solution of a number of problems. Thus in optimal exploitation of fisheries, we consider

(i) use of dynamic programming, calculus of variations and maximum principle to find the maximum value of the present value of total profits over an infinite span of time for a single species

(ii) use of same techniques for the optimal exploitation of multispecies communities

(iii) obtaining of limiting optimal solutions when overall optimal solutions are not obtainable.

In optimal exploitation of mines, we consider two problems

(i) optimal exploitation when time span is fixed

(ii) optimal exploitation when time span is variable and can be chosen so as to maximize the maximum present value of the profit for variation in length of time span.

In linear birth-death-immigration-emigration processes, we consider

(i) linear birth-death-emigration process (unsteady case)

(ii) discrete birth-death-immigration-emigration process.

In epidemic theory, we consider the spread of epidemic for the case of multiple infections and study

(i) deterministic model

(ii) stochastic model

Each of these areas has a large literature including dozens of standard level monographs and hundreds of research papers. In the next four sections we give a brief survey of that part of the literature which is related to our work in the present thesis in order to give a perspective for the work we have done. We also give a very brief summary of the results obtained in all areas. Chapter II - V will give detailed derivation of the results.

Most of the work reported in the thesis has been published or accepted for publication or is ready for communication in the form of the following papers :

Optimal exploitation of controlled fishers, to appear in J. Math. Physi. Sci.

Optimal exploitation of multispecies fisheries, published in Some Aspects of Mechanics of Continua, part 1, 1977.

Limiting optimal solutions for exploitation of fisheries, to appear in Proc. Nat. Acad. Sci.

A Mathematical model for optimal exploitation of mines, to appear in J. Math. Physi. Sci.

A class of birth-death-emigration processes, pub. in Nat. Acad. Sci. letters, Vol. 2, 1, 1979.

Discrete birth-death-emigration-immigration processes, to be communicated.

1.2. OPTIMAL EXPLOITATION OF RENEWABLE RESOURCES : FISHERIES

1.2.1 SURVEY OF LITERATURE

Renewable resources are those capable of regeneration like fish, timber etc. Clark [1976] states : " they grow by the gift of nature."

The most important feature of a renewable resource also called replenishable or reproducible resource is its law of biological growth. This characteristic can be studied in many ways e.g. as growth in the number of members vide Berverton and Hott [1957], or as growth in the total biomass of the species vide Lotka [1956].

Extremely simple mathematical models for the exploitation of biological resources (renewable resources) are given by the ordinary differential equation of the form [Clark [1976]]

$$\frac{dx}{dt} = f(x) - h(t),$$

where

$x = x(t)$ denotes the size of the population or biomass at time t.

$f(x)$ = a given function defining the natural growth rate of population.

and $h(t)$ = rate of removal or harvesting.

The management of renewable resources had been based on the concept of maximum sustainable yield. This concept was based on the assumption that at any population level, less than a certain level, a surplus of production existed which could be harvested without disturbing the stock level. The models exhibiting this trend of growth were studied by Gordon [1954], Shaefer [1957], Plourde [1970] and Christy [1973]. But this concept had to face several objections raised on both biological and socio-economic grounds. Different forms of growth functions have been reported by May [1973] and Clark [1976].

The subject of an economic theory, developed in the literature since Gordon [1953], has been the economic effects of uncontrolled exploitation of a common property resource. Gordon [1954] and Scott [1955] have made contributions to the common property resources. Such resources where ownership is not perfect, are often over exploited since they allow the entry of a surplus of labour and capital into the industry.

In his articles Gordon [1953,54] argued that this entry would continue untill all rent of fishery was dissipated vide Clark [1976]. A theory exists in support of the argument that this type of exploitation would lead to the dissipation of profits and ultimately depletion of the resources.

Thus for policy makers the problem has been to device the methods to correct this misallocation of resources. Apart from other ways, the greater control of exploitation has been done by putting restrictions on quotas and efforts. Remaining in the domain of quota or effort restrictions, fisheries exploitation can be managed to attain various objectives which may be economic, social or both.

A number of contributions to the economic analysis of the exploitation of fisheries have been due to many including Smith [1968], Scott [1955], Clark [1971, 72, 73], Beddington, Watts and Wright [1975], Waugh and Calvo [1974] and Colbert and Swan [1975]. Smith [1968] had provided a unified general model which with simplifying assumptions could be applicable to a wide range of resource problems. This general model could not make out the detailed investigations of question of extinction as well as levels and stability of equilibriums. Particular questions concerning the behaviour of the population as it nears zero or critical levels of exploitation were considered in detail by Clark [1971, 72]. Clark [1973] studied a mathematical model for the Antartic

blue whale with constant price and a particular function for total costs. Colbert and Swan [1975] made good replacement of weak assumptions of Clark [1973] regarding price and cost function.

Clark [1971,72] had taken constant costs as well as given prices which prevented him from assessing the effects of price elasticities and of different characteristics of production function. Beddington, Walts and Wright [1975] had studied the consequences of different forms of production functions. The object was to maximize the present value of total profits subject to a production function and two differential equations defining the rate of change of capital stock and population. Waugh and Calvo [1974] saw the fisheries as exhaustible resources with the same object.

Most of the contributions to the economic theory of exploitation of fisheries has been through the maximization of some objective function which is economic in nature. A very commonly used objective function is the present value of total profits over time. This is maximized subject to a differential equation representing the rate of change of population along with a restriction on quota.

Sancho and Mitchell [1975] used dynamic programming technique to obtain the optimal effort each year to optimize the present value of total profits over time. Sancho and Mitchell [1977] applied the model to the Canadian ground fish

industry and using the data over ten-year period, obtained the optimal effort as well as optimal present value of total profits. Silvert and Smith [1977] considered the maximization of discounted present value of net Revenue given by a function of vector population and vector harvest while studying the exploitation of multispecies community.

1.2.2 DESCRIPTION OF PROBLEMS AND RESULTS

Sancho and Mitchell [1975] used dynamic programming to obtain optimal effort each year to optimize the present value of total profits over time. Subject to a differential equation defining the rate of change of fish population and a constraint on effort in order to take into account the specified quota that may be alloted to the fishing party or the capacity of the market to buy fish. Sancho and Mitchell [1975] found their results in terms of R, the initial population. Thus if the maximum value of total profits is needed each year, we have to make the estimate of R each year which can be done heuristically by taking the ratio of catch to the effort and comparing one with the next.

In our first problem of optimal exploitation of controlled fisheries, firstly we study the model of Sancho and Mitchell [1975] by using Maximum principle and Calculus of variations. Then we impose the constraint on catch per unit time. Finally, considering exploitation over finite horizon of time the following results are found out.

We are able to obtain our results explicitly as functions of time and thus avoid the heuristic approach, which is necessary in Sancho and Mitchell [1975] if the maximum value of total profits is needed each year.

We can also find out the conditions under which the solutions are optimal.

We can also find out the conditions which are to be satisfied if we impose the constraint on catch per unit time.

Exploitation over finite horizon of time does not requires the above conditions.

In the second problem, we use optimality principle of dynamic programming and the maximum principle to optimize the present value of total profits (P.V) over all time in a fishery exploitation when the fishery has a number of different but non-interacting n species. This is because the area of the pond or a river or a sea may contain a number of different types of fish commanding different prices and all of them may be caught. We get the following results.

The maximum value of present value of profits is a quadratic function of $R_1, R_2, \ldots, R_n$, the initial fish populations. As an example, a particular case of two species is solved completely.

The case when the catch of any species per unit time depends only on its own population is solved by using

maximum principle still getting present value as a quadratic function of $R_1, R_2, \ldots, R_n$. Some more results for the limiting case are also pointed out.

In the last problem, it is shown that limiting optimal solutions may be possible for an ecological system which may not, otherwise, be tractable. We consider a general system. As a special case the system considered by Clark [1976] is solved completely for optimal equilibrium solutions. The case for interacting species is also considered. We observe

When we are not able to find the optimal non-equilibrium singular controls, we find the equilibrium optimal solutions. If initial populations are less than the equilibrium values we don't harvest till populations rise to equilibrium values. If either or both initial populations are more than equilibrium values, we do maximum harvesting possible required by the market till equilibrium values are reached. After equilibrium values are reached, we use the optimal effort (limiting value) thereafter and the populations remain stable.

1.3 OPTIMAL EXPLOITATION OF EXHAUSTIBLE RESOURCES : MINES

1.3.1 SURVEY OF LITERATURE

Exhaustible resources are those not capable of regeneration, such as petroleum, natural gas and minerals etc. They are fixed assets. As industry has developed, the need for

minerals has increased and the search for more minerals has gone on. Today, the mineral resources of the earth are so commonly used that they affect every aspect of our lives.

Minerals have some distinct characteristics which distinguish them from other natural resources. According to Erich Zimmerman, "the most disconcerting feature of minerals is their exhaustibility." By and large, minerals are depleting assets and once mined out the deposits get depleted without any chance of replenishment.

The mathematical treatment of the subject goes back to Hotelling [1931] who emphasized the necessity of employing the methods of calculus of variations in the theory of exhaustible resources. The earlier works by Gray [1914] and Hotelling [1931] concerning the "The theory of mine" have been extended by many including Smith [1968], Waugh and Calvo [1974], Gordon [1967], Goldsmith [1974] and Clark [1976].

But in all these studies, while dealing with exhaustible resources, the effort that we apply while extracting the amount of mineral per unit time has not been taken into account. This aspect is studied in the third chapter by assuming different extraction functions and with some cost function as also functions of effort applied per unit time.

1.3.2 DESCRIPTION OF PROBLEMS AND RESULTS

In all the problems, our aim is to maximize the present value of total profits over time (P.V). The main

difference from the renewable resources is that the mine is to be exhausted in a finite time-interval. We deal with two types of problems, in the first we maximize present value of total profits so that the mine is exhausted in a given time-interval and in the second we choose that time-interval for exploitation which provides us with the maximum of all the maximum P.Vs. Taking cost function quadratic in effort [Sancho and Mitchell [1975]] the procedure is illustrated numerically in all the problems which are given below along with results.

Firstly, a simple mathematical model is used in which the extraction function is linear in amount of material and the effort. Calculus of variations and Maximum principle are used to obtain the amount of the material to be taken out and the optimal effort for getting the maximum of the P.V. over the time-period in which the mine is exhausted. We observe the following

> For a fixed set of values of parameters the maximum profit increases with time and finally stabilizes showing that in very large time we do not get any additional profit. Graphs for amount of material and optimal effort against time are also plotted.

Secondly, we consider the extraction function as quadratic in amount of material and as linear in effort. Using calculus of variations, we deal with the same problem as above. Giving a numerical example, the following result is observed.

> Slower exploitation of a mine over a long period is more profitable than quicker exploitation over a short time. The variations of amount of material, the optimal effort and the rate of change of the amount of material with time are shown in different graphs.

Lastly, we consider the problem in which the extraction function is some product of the amount of material in the mine and the effort. Using calculus of variations we are again faced with the similar problem as above. The result is as follows :

> Even in this case slow exploitation is preferred to quicker exploitation since the latter is more profitable. Graphs show the variation of the amount of material and the effort with time.

1.4 LINEAR BIRTH-DEATH-IMMIGRATION-EMIGRATION PROCESSES

1.4.1 SURVEY OF LITERATURE

Early studies in the mathematical theory of population growth, epidemics, queues and several other situations were primarily concerned with the development of deterministic models which determine the number of individuals in the system at any time t. In this approach, on the basis of some assumptions concerning changes in numbers of individuals in the system in a small interval of time, an equation (differential or integral, etc.) for the number of individuals in the system, say $x(t)$, is

formed. This equation is solved subject to some initial condition to find out x(t). By assuming x(t) as a real valued continuous function of time and the change in the small interval of time as deterministic, one obtains the result that number of individuals in the system does not change if initial conditions are not altered. Therefore, this approach does not take into consideration the large number of random or chance factors that can influence the system.

In 1939, birth and death processes were introduced by W. Feller. These processes have since been used to describe stochastic models for population growth, queues, epidemics and telephone calls etc. In this approach, a random variable X(t), integer (or real) valued, is used to describe the size of the system at time t and the change in the system in a small interval of time is expressed in terms of probabilities of certain elementary events in that interval. Let P(n,t) denote the probability of there being n persons in the system at time t i.e.

$$P(n,t) = P[X(t) = n] \quad , \quad n = 0,1,2,3,\ldots$$

In birth-and-death processes, a system of differential-difference equations in P(n,t) is formulated. Then it is solved with some initial conditions to find out the size of the system at any time t.

When the population size is n, let the probability of a birth, a death, an immigration and an emigration in time

interval $(t, t + \Delta t)$ be given respectively by $\lambda_n \Delta t + O(\Delta t)$, $\mu_n \Delta t + O(\Delta t)$, $\nu_n \Delta t + O(\Delta t)$ and $\alpha_n \Delta t + O(\Delta t)$.

Depending upon the nature of the probability function, the following processes have been classified in the literature.

(i)	Poisson Process	when	$\lambda_n = \lambda$, $\mu_n = 0$, $\nu_n = 0$ and $\alpha_n = 0$
(ii)	Pure Birth Process	"	$\lambda_n = \lambda_n$, $\mu_n = 0$, $\nu_n = 0$ and $\alpha_n = 0$
(iii)	Pure Death Process	"	$\lambda_n = 0$, $\mu_n = \mu_n$, $\nu_n = 0$ and $\alpha_n = 0$
(iv)	Linear Birth-Death Process	"	$\lambda_n = n\lambda$, $\mu_n = n\mu$, $\nu_n = 0$ and $\alpha_n = 0$
(v)	Non-Homogenous Birth-Death Process	"	$\lambda_n=\lambda_n(t)$, $\mu_n=\mu_n(t)$, $\nu_n=0$ and $\alpha_n=0$
(vi)	Linear Birth-Death-Immigration Process	"	$\lambda_n=n\lambda$, $\mu_n=n\mu$, $\nu_n = \nu$ and $\alpha_n = 0$
(vii)	Linear Birth-Death-Immigration-Emigration Process	"	$\lambda_n=n\lambda$, $\mu_n=n\mu$, $\nu_n = \nu$ and $\alpha_n = \alpha$.

The complete solutions for the first six cases (when $\alpha_n = 0$) have been given, among others, by Bailey [1964], Bartlett [1955], Bharucha-Reid [1960], Feller [1959], Iosifescu and Tautu [1973], Kendall [1948, 1952] and Srinivasan and Mehta [1976].

Recently Getz [1975, 1976] attempted to discuss the case (vii) when all the four processes are involved, but he was led to a deceptively simple treatment by omitting a term of probability of extinction. This error was pointed out by Kapur [1979a]. Kapur [1979b, 1979c] also found out the

complete solution in terms of incomplete beta functions when three processes i.e. birth, death and emigration are involved. Kapur [1978a, 78b, 78c, 78d, 79d, 79f] and Kapur and Kapur [1978] also obtained the complete solution for the steady case in terms of hypergeometric and generalized hypergeometric functions.

1.4.2 DESCRIPTION OF PROBLEMS AND RESULTS

While dealing with the birth-death-immigration-emigration process Getz [1975, 1976] found a partial differential equation for generating function in which he omitted one term containing the probability of extinction. This omission makes all his calculations wrong. Knowing the probability of extinction is the foremost necessity if we want to know all about the system. Kapur [1979b, 1979c] calculated the probability of extinction of B-D-E process in terms of incomplete beta functions. We give here an alternative method for finding the probability of extinction. We put our problem as follows.

The probability of extinction for a birth-death-emigration process for the case ($\alpha < \mu$, emigration rate is less than the death rate) is obtained in terms of the solution of an Abel's integral equation which is solved numerically. We observe

As expected the agreement between our solution and that of Kapur [1979b, 1979c] is quite good.

In another problem, we consider the discrete version of the continuous-time birth-death-immigration emigration process

and solve it numerically. The result is stated as follows

We get more insight into the system and draw some general conclusions. Different probabilities have been plotted for $n_o = 3$ and $n_o = 6$, where n_o is the initial number of persons in the system against time as well as against number of persons. Graphs for Mean and variance have also been plotted.

1.5 SPREAD OF EPIDEMICS

1.5.1 SURVEY OF LITERATURE

For the literature in the theory of epidemics, upto 1957, an excellent monograph is by Bailey [1957]. A comprehensive study of the mathematical contribution to the description of the spread of epidemics has been done by Dietz [1967] in the form of an excellent review.

In recent years, much of the work in epidemics is related to the control problems studied by many including Bobisud [1977], Wickwire [1976] and Jaquette [1970] but we will not deal with these types of problems.

Earlier, in the theory of epidemics, the disease was supposed to be spread by infected persons, called injectives, who were recognised by their having some symptoms. But Weiss [1965] introduced a new aspect in the spread of epidemics i.e. spread by carriers. A carrier is defined to be an individual who does not have overt disease symptom but who is however able to communicate the disease to others. Other contributions in

this field are due to Downton [1968] and Abakuks [1974]. We will not also discuss this aspect .

A widely used model in epidemics assumes the division of any fixed population size into three categories (i) susceptibles : those individuals who can catch hold of infection when an infected person comes into contact with them (ii) Infectives : those individuals who are capable of communicating the disease to others and (iii) Removed : those individuals who have been removed either by isolation, death or permanent immunity.

Two types of models, deterministic and stochastic, have been investigated by researchers. Deterministic models which are expressed in terms of systems of differential equations for the numbers in each categories have been applied since the work of Hamer [1906]. But since these models do not account for the considerable variations which so often occur in epidemic situations, stochastic models are much more suitable for this kind of problem. Except for the case when the number of both susceptibles and infectives is quite large, in all other situations there may be important differences in the results predicted by both versions.

Generally two types of situations have been studied in the theory of epidemics (i) simple epidemic (ii) General epidemic. Deterministic and stochastic models have been applied in both situations.

Simple epidemic is one in which infection spreads by contact between the members of a community and no removal is

possible either by death, recovery or isolation. Therefore all susceptibles become infected. The contributions to this type of situation have been made by many. Among them are Bailey [1959, 1963, 1967a], Kendall [1957], Severo [1967a], Williams [1965a], Hill and Severo [1969], Hethcote [1973] and McNeil [1972]. Bailey [1950, 1957] has given a solution for the state probabilities $p_r(t)$, where r denotes the number of susceptibles at any time t with a single infective initially. Later in his paper [1963] he obtained an expression for the generating function to calculate the state probabilities in a simple way. Haskey [1954] obtained a general expression for mean number of susceptibles which was derived by Bailey [1963]. Williams [1965] considered the mean and variance of susceptibles with one or more infectives and obtained the results for large populations. The case with small populations and one or more initial infectives has been considered by Hill and Severo [1969]. Hethcote [1973] studied the asymptotic behavior of a simple deterministic model in which infectives become susceptibles as soon as they are removed.

The general epidemic admits removal of infectives from circulation, either by isolation, death or immunity. In any case, removals may not infect other susceptibles. Though McKendrick [1926] and Bartlett [1949] had already studied this situation, a time-dependent solution was given by Gani [1965] and Siskind [1965]. There are some new results about the total size of epidemic by Daniels [1967], Downton [1967a]

and Siskind [1965]. Billard [1965] considered the case when an infective after recovery again behaves as a susceptible and that this recovery may take place a finite number of steps. Severo [1969, 1969b] considers generalizations of some slochastic models and studies the case allowing for removal of susceptibles as well as of infectives.

1.5.2 DESCRIPTION OF PROBLEMS AND RESULTS

But in all these studies, the problem of single infection has been considered in a time interval $(t, t + \Delta t)$. Even in birth and death processes, most of the studies are concerned with single birth and single death in the time interval $(t, t + \Delta t)$ i.e. it is assumed that the probability of more than one infection in this time-interval is of the order $(\Delta t)^2$. We consider the case when multiple infections can occur even when a single infective comes into contact with a number of susceptibles at a time. The corresponding case of multiple birth in a linear birth-death process has already been discussed by Doubleday [1973]. In chapter V we try to get insight into multiple-infection situation by considering typical cases with small populations.

We consider in chapter V a general case in which m infections can occur at a time when a infectives come into contact with n susceptibles initially. For simple epidemic we draw the following conclusions

For $m \leq n/2$ when n is even or $m \leq n-1/2$ when n is odd, we get a system of differential-difference equations

in which the first m and the last m equations are written separately. But the middle n+1 - 2m equations can be written as one equation with r varying from m to n-m. Otherwise, we have to write all n+1 equations separately. Solving the system for $n = 3$, $m = 2$ and $n = 6$, $m = 2$ we observe that multiple infection causes the process of infection to be faster and it takes little time for all the susceptibles to be infected. The initial probabilities tend to zero while later probabilities tend to 1 more quickly. Epidemic curves, mean and variance are also compared. For general epidemics, we observe that We write a general differential-difference equation in r and s, the number of susceptibles and infectives respectively. Taking into account the restrictions on r and s and considering the probability with negative suffix as zero, all $(n+1)(n+a+1) - \frac{n(n+1)}{2}$ non-zero probabilities can be obtained. As an example a complete system of differential-difference equation is presented here for $n = 3$, $a = 2$ and $m = 2$ and is solved numerically by using Gill's method of integration.

We also plotted the following graphs :

(i) All probabilities of extinction for susceptibles.

(ii) $p_{na}(\tau)$, the probability that no susceptible gets infected and no infective is removed.

(iii) $p_{oo}(\tau)$, the probability that all susceptibles get infected and all infectives are removed.

CHAPTER II

OPTIMAL EXPLOITATION OF CONTROLLED FISHERIES

2.1 INTRODUCTION Since exploitation of common property fisheries had led to serious consequences even including the complete depletion of the resources [Gordon [1953, 1954]], great attention has been paid, in the economic theory of exploitation of fisheries, to some sort of control so that the fishery resource can be prevented from vanishing. This control has been applied mainly through the restrictions on effort or quota along with the maximization of some objectives which may be economic, social or both.

Maximization of present-value of total profits over time has been sought by many including Beddington, Watts and Wright [1975], Waugh and Calvo [1974], Sancho and Mitchell [1975, 77], Silvert and Smith [1977] and Clark [1976].

This chapter is divided into three more sections namely Sections 2.2, 2.3 and 2.4 in which we study

(i) Sancho and Mitchell Model

(ii) Multispecies fisheries model

and (iii) Mathematically non-tractable models in fisheries for which only limiting control solutions can be obtained.

Following Sancho and Mitchell [1975] and a few others our object is also to maximize the present-value of total profits

over time subject to constraint on fishing effort or catch, by the use of dynamic programming or maximum principle or calculus of variations in all the three sections. The necessity for each of this type of study is pointed out in the following paragraphs.

Section 2.2 Sancho and Mitchell [1975] used dynamic programming to obtain the optimal effort each year so as to maximize the present-value of total profits over time with the restriction on the maximum effort that can be applied. Sancho and Mitchell [1975] observe that " The results are obtained in terms of R, the fish population at time $t = 0$. If the maximum value of total profits is required each year, then the estimate of R would have to be made each year. Heuristically, this can be done by taking the ratio of catch to effort each year and comparing one with the next."

We show that the heuristic approach is not necessary when we use maximum principle and calculus of variations. We get optimal effort and optimal population as functions of time.

As Pontecorvo [1967] states : " Regulations specifying the size of capture are far more important than regulations determining the optimum level of effort." We impose the constraint on fishing party not directly in terms of effort it can make, but in terms of quota it can harvest per unit time.

We also find out the conditions under which our optimal solutions exist. This is not possible in the method adopted by Sancho and Mitchell [1975].

Section 2.3. In this section we include the aspect that the area of pond or a river or a sea may contain a number of different types of fish commanding different prices and all of them may be caught.

Section 2.4. In this section it is shown that limiting optimal solutions may be possible for an ecological system which may not, otherwise, be mathematically tractable.

2.2 SANCHO AND MITCHELL MODEL

Sancho and Mitchell [1975] have used dynamic programming to find optimal effort each year to optimize the present value of the total profits over all time. In this section we use maximum principle and calculus of variations to obtain additional results.

2.2.1 THE MODEL. The standard model used by Sancho and Mitchell [1975], for fisheries exploitation is

$$\frac{dX}{dt} = aX - C(t) \tag{1}$$

where $X(t)$ = fish population density or biomass at time t.

a = natural rate of growth of fish

and $C(t)$ = catch of fish taken per unit time at time t.

Bell and Carlson [1970] took $C(t)$ as

$$C(t) = \alpha + \beta X(t) + \gamma E(t), \tag{2}$$

where α, β, γ are parametric constants and $E(t)$ is the fishing effort per unit time at time t. Using (1) and (2), we get

$$\frac{dX}{dt} = (a - \beta)X - \alpha - \gamma E(t) \qquad (3)$$

The present value P of the profits is given by

$$P = \int_0^\infty e^{-\lambda t} [pC(t) - q(E)] \, dt, \qquad (4)$$

where $\lambda > 0$, represents the given instantaneous discount rate, p is the fixed price per unit of the fish harvested and q(E) is the cost of the fishing effort. In Clarks, Edwards and Friedlander [1973], q(E) was taken as a linear function of E, while Sancho and Mitchell [1975], following Holt, Modigliani, Muth and Simon [1960], took it as a quadratic function of E so that their mathematical problem was to maximize

$$P = \int_0^\infty e^{-\lambda t} [p(\alpha + \beta X + \gamma E) - (bE^2+kE+\ell)] \, dt, \qquad (5)$$

subject to (3) and

$$E(t) \leq Q, \qquad (6)$$

where Q is the constraint on fishing effort in order to take into account the specified quota that may be allotted to the fishing party or the capacity of the market to buy fish. Using dynamic programming, they solve the problem obtaining the maximum value of P as

$$f(R) = AR^2 + BR + K, \qquad (7)$$

where $X(0) = R$ is the initial population.

In the special case when $k = 0$ and $\ell = 0$, they get

$$A = \frac{(\lambda - 2a + 2\beta)b}{\gamma^2} \tag{8}$$

$$B = \frac{p\beta + (\lambda - 2a + 2\beta)(p - \frac{2b\alpha}{\gamma^2})}{2\lambda - 3a + 3\beta} \tag{9}$$

and $$K = \frac{1}{\lambda} \left[p\alpha + \frac{1}{4b} (p - B)^2 \gamma^2 - \alpha\beta \right] \tag{10}$$

They also obtain the optimal effort E in this case as

$$E = \frac{\gamma}{2b} \left[p - 2AR - B \right] \tag{11}$$

2.2.2 DISCUSSION OF EARLIER RESULTS

Sancho and Mitchel [1975] found their results in terms of initial population R. Taking the ratio of catch to to effort each year and comparying each year with the next year's, the estimate of R would have to be made each year, if the optimal policy is to be always followed. We show that the heuristic approach is not necessary since we can find for the optimal policy, both X(t) and E(t) explicitly as functions of time.

Moreover the constraint that is imposed on a fishing party is not directly in terms of the effort it can make, but in terms of the quota it can harvest per unit time so that we replace (6) by

$$C(t) = \alpha + \beta X(t) + \gamma E(t) \leq Q. \tag{12}$$

Initially this would imply

$$\alpha + \beta R + \gamma E \leq Q. \tag{13}$$

If (11) does not satisfy this condition, then formula (7) also fails to hold and there is need for an alternate formula.

The dynamic programming formulation also proceeds on the assumption that the optimal solution exists and as such the conditions under which (7) and (11) give the solution are not brought out in Sancho and Mitchell [1975]. We discuss the conditions in the next section.

2.2.3 USE OF MAXIMUM PRINCIPLE

To maximize P given by (5) subject to (3) and (12), we form the Hamiltonian [Pontreagin [1960], Clark [1976]]

$$H(t,X,E,\psi) = e^{-\lambda t}\{p(\alpha+\beta X+\gamma E) - (bE^2+kE+\ell)\}$$

$$+ \psi(t)\{(a-\beta)X-\alpha - \gamma E\}, \tag{14}$$

where $\psi(t)$ is an auxiliary function satisfying

$$\frac{\partial H}{\partial X} = - \frac{d\psi}{dt} \tag{15}$$

We then maximize H as a function of E to get

$$e^{-\lambda t}(p\gamma - 2bE - k) - \gamma\psi(t) = 0 \tag{16}$$

We shall take into account the constraint (12) later. (Sec. 2.2.7). From equation (15), we get

$$e^{-\lambda t} p\beta + (a - \beta)\psi(t) = - \frac{d\psi}{dt} \tag{17}$$

Integrating (17), we get

$$\psi(t) = \frac{p\beta}{(\lambda - a+\beta)} e^{-\lambda t} + A_1 \bar{e}^{(a-\beta)t}, \tag{18}$$

where A_1 is an arbitrary constant. Substituting in (16) we get

$$E(t) = K_1 - K_2\, e^{(\lambda-a+\beta)t}, \tag{19}$$

where
$$K_1 = \frac{(p\gamma - k)\,(\lambda-a+\beta) - \gamma p\beta}{2b(\lambda-a+\beta)} \tag{20}$$

and
$$K_2 = \frac{A_1\gamma}{2b} \tag{21}$$

Substituting in (3) and integrating, we get

$$X(t) = \frac{\alpha+\gamma K_1}{a-\beta} + \frac{\gamma K_2}{(\lambda-2a+2\beta)}\, e^{(\lambda-a+\beta)t} + A_2\, e^{(a-\beta)t} \tag{22}$$

where A_2 is an arbitrary constant. Substituting in (5) from (22) and (19) we can get the maximum value of P.

2.2.4 DISCUSSION OF SOLUTION

We find from (3) that if $a < \beta$, then $\frac{dX}{dt}$ is always negative and X(t) would go on decreasing and the fishery would soon be exhausted.

Moreover if $a < \beta$, then $\lambda - a + \beta > 0$ and the second term on the R.H.S. of (22) would tend to infinity as t increases. If the population becomes large, our simple model

$$\frac{dX}{dt} = aX \tag{23}$$

for the growth of the fish in the absence of exploitation would not be valid. Moreover when we substitute (22) and (19) in (5), we get a term

$$\int_0^\infty e^{(-a+\beta)t}\,dt,$$

which tends to infinity if $a < \beta$. To avoid this we will have to choose $K_2 = 0$ or $A_1 = 0$, so that $E(t)$ would be constant and $X(t)$ would approach $\frac{\alpha + \gamma K_1}{a-\beta}$ which can be negative for some values of parameters.

To avoid all these complications, we shall assume

$$a > \beta \tag{24}$$

In order to prevent the population from going to infinity, we shall further take

$$A_2 = 0 \text{ and } \lambda - a + \beta < 0 \tag{25}$$

Using the initial condition $X(0) = R$, (22) then gives

$$R = \frac{\alpha + \gamma K_1}{a-\beta} + \frac{\gamma K_2}{(\lambda - 2a + 2\beta)}, \tag{26}$$

so that

$$X(t) = \frac{\alpha + \gamma K_1}{a-\beta} + \left(R - \frac{\alpha + \gamma K_1}{a-\beta}\right) e^{(\lambda-a+\beta)t} \tag{27}$$

and (19) then gives

$$E(t) = K_1 - \frac{(\lambda-2a+2\beta)}{\gamma}\left(R - \frac{\alpha + \gamma K_1}{a-\beta}\right) e^{(\lambda-a+\beta)t} \tag{28}$$

As $t \to \infty$

$$X(t) \to \frac{\alpha + \gamma K_1}{a-\beta}, \quad E(t) \to K_1 \tag{29}$$

We shall assume $p\gamma > k$, then from (20) and (25), $K_1 > 0$. If $R > \frac{\alpha + \gamma K_1}{(a-\beta)}$, the fish population decreases to its equilibrium

value if $R < \frac{\alpha + \dot{\gamma} K_1}{(a-\beta)}$, the fish population increases to its equilibrium value.

For evaluating (5), we have to find the integrals

$$\int_0^\infty e^{-\lambda t} X(t)\, dt, \quad \int_0^\infty e^{-\lambda t} E(t)\, dt, \quad \int_0^\infty e^{-\lambda t} E^2(t)\, dt$$

The last integral would be finite only if

$$\lambda - 2a + 2\beta < 0 \tag{30}$$

which is true from (25).

We find from (27) and (28) that the behaviour of E(t) and X(t) is the same as $t \to \infty$ i.e. if $R > \frac{\alpha + \gamma K_1}{(a-\beta)}$, both X(t) and E(t) decrease to the equilibrium values and if $R < \frac{(\alpha + \gamma K_1)}{(a-\beta)}$, then both X(t) and E(t) increase to the equilibrium values. This has been shown in figures 2.1, 2.2, 2.3 and 2.4.

2.2.5 COMPARISON WITH EARLIER RESULTS

Substituting from (27) and (28) in (5), integrating, using (25) and (30) and simplifying, we easily find the maximum value of P to be given by the quadratic expression (7) with the difference that when $k = 0$ and $\ell = 0$, B has to be replaced by

$$B' = \frac{p\beta - (\lambda - 2a + 2\beta)\left(p + \frac{2b\alpha}{\gamma^2}\right)}{(a-\beta)} \tag{31}$$

This difference is seen to be due to a calculation error in Sancho and Mitchell [1975] where equation (20) should be

$$\lambda B = p\beta - \frac{A\gamma^2 (p - \beta)}{b} + (a-\beta)B - 2A\alpha \tag{32}$$

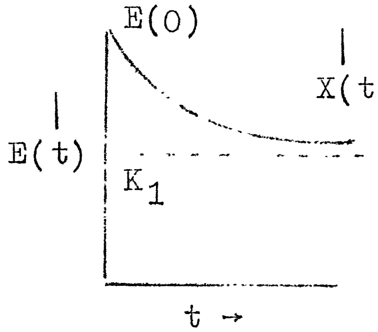


Fig. 2.1

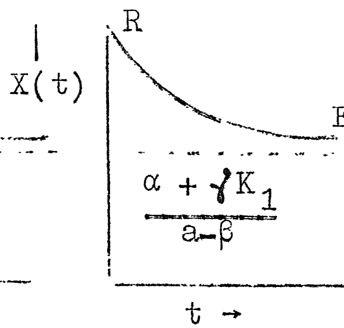


Fig. 2.2

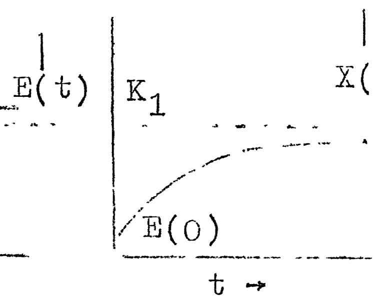


Fig. 2.3

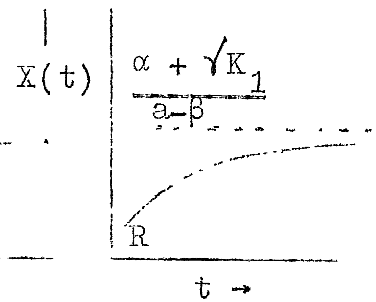


Fig. 2.4

The second term on the R.H.S. has a negative sign and not the positive sign as given there. Also from (28)

$$E(0) = K_1 - \frac{\lambda - 2a + 2\beta}{\gamma} \left(R - \frac{\alpha + \gamma K_1}{a-\beta}\right) \tag{33}$$

which is the same as (11) when $k = 0$ and modified value in (31) is used.

The expressions for the maximum value of P and the effort required initially are the same as those of Sancho and Mitchell [1975] when minor error in B is corrected. However we have also obtained $E(t)$ and $X(t)$ and we have not to use heuristic methods.

We have also obtained the conditions

$$a > \beta, \ \lambda < a-\beta \tag{34}$$

under which our results are valid.

From (27) and (28) we obtain

$$E(t) = K_1 - \frac{\lambda - 2a + 2\beta}{\gamma} \left(X(t) - \frac{\alpha + \gamma K_1}{a-\beta}\right) \tag{35}$$

(33) is a special case of (35) when $t = 0$, i.e. the effort to be used at any time depends on the population at that time in the same form as the initial effort depends on the initial population.

This is also expected. Figure 2.5 also shows that if $R \gtrless \frac{\alpha + \gamma K_1}{a-\beta}$, then $E \gtrless K_1$. The slope of the straight line is positive since $\lambda - 2a + 2\beta < 0$.

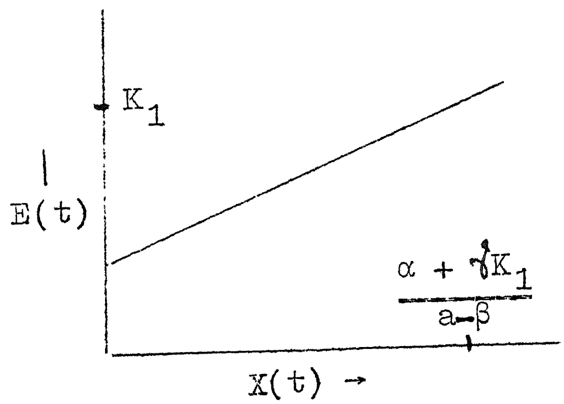


Fig. 2.5

It can also be varified that the maximum value of

$$\int_T^\infty e^{-\lambda t} \{ p(\alpha + \beta X + \gamma E) - bE^2 \} \, dt \qquad (36)$$

is $AX^2(T) + B'X(T) + K$, as expected. (37)

2.2.6 USE OF CALCULUS OF VARIATIONS

Substituting from (3) for $\dot{E}(t)$ in (5), we get

$$P = \int_0^\infty e^{-\lambda t} \Big[p\alpha + p\beta X + \frac{(p\gamma - k)}{\gamma} \{(a-\beta)X - \alpha - \dot{X}\}$$

$$- \frac{b}{\gamma^2} \{(a-\beta)X - \alpha - \dot{X}\}^2 - \ell \Big] \, dt \qquad (38)$$

Using Euler-Lagrang's Equations of calculus of variations and simplifying, we get

$$\ddot{X} - \lambda\dot{X} + (a-\beta)(\lambda - a+\beta) X = L_1, \qquad (39)$$

where

$$L_1 = (\lambda - a+\beta) \Big[\frac{(p\gamma - k)\gamma}{2b} + \alpha \Big] - \frac{p\beta\gamma^2}{2b} \qquad (40)$$

Solving (39), we get

$$X(t) = B_1 \, e^{(\lambda - a+\beta)t} + B_2 \, e^{(a-\beta)t} + \frac{L_1}{(a-\beta)(\lambda - a+\beta)}, \qquad (41)$$

where B_1 and B_2 are arbitrary constants.

Now from (20) and (40)

$$\frac{L_1}{(\lambda - a+\beta)} = \alpha + \frac{(p\gamma - k)\gamma}{2b} - \frac{p\beta\gamma^2}{2b\,(\lambda - a+\beta)}$$

$$= \alpha + \gamma \, \frac{(p\gamma - k)(\lambda - a+\beta) - p\beta\gamma}{2b(\lambda - a+\beta)} = \alpha + \gamma K_1, \qquad (42)$$

so that

$$X(t) = B_1 e^{(\lambda - a+\beta)t} + B_2 e^{(a-\beta)t} + \frac{\alpha + \gamma K_1}{(a-\beta)} \qquad (43)$$

Since $a - \beta > 0$, and we don't want $X(t)$ to increase indefinitely as $t \to \infty$, and $X(0) = R$, (43) gives

$$X(t) = \frac{\alpha + \gamma K_1}{a-\beta} + \left(R - \frac{\alpha + \gamma K_1}{a-\beta}\right) e^{(\lambda - a+\beta)t}, \qquad (44)$$

which is same as (27).

Substituting in (3) we get (28). Thus calculus of variations gives the same results as the maximum principle gives.

2.2.7 <u>THE EFFECT OF CONTROLLED QUOTAS</u>

If there is a quota Q fixed for the party or if the market demand $\leq Q$, then from (12), (27) and (28), we require

$$\alpha + \beta \left\{ \frac{\alpha + \gamma K_1}{a-\beta} + \left(R - \frac{\alpha + \gamma K_1}{a-\beta}\right) e^{(\lambda - a+\beta)t} \right\}$$

$$+ \gamma \left\{ K_1 - \frac{(\lambda - 2a + 2\beta)}{\gamma} \left(R - \frac{\alpha + \gamma K_1}{a-\beta}\right) e^{(\lambda - a+\beta)t} \right\} \leq Q$$

which gives

$$\frac{a}{a-\beta}(\alpha + \gamma K_1) + \left(R - \frac{\alpha + \gamma K_1}{a-\beta}\right)(2a - \lambda - \beta) e^{(\lambda - a+\beta)t} \leq Q \qquad (45)$$

If (45) is satisfied for all values of t i.e. if the catch finally and initially given by

$$Y = \frac{a}{a-\beta}(\alpha + \gamma K_1) \text{ and } Z = aR + \left(R - \frac{\alpha + \gamma K_1}{a-\beta}\right)(a - \lambda - \beta) \qquad (46)$$

is $\leq Q$, then the solutions (27) and (28) give optimal functions.

If $R \geq \dfrac{\alpha + \gamma K_1}{a-\beta}$, then $Z \leq Q \implies Y \leq Q$.

If $R \leq \dfrac{\alpha + \gamma K_1}{a-\beta}$, then $Y \leq Q \implies Z \leq Q$.

Thus the conditions for solutions (27) and (28) to be optimal are

$$\text{(i)} \quad R > \frac{\alpha + \gamma K_1}{a-\beta}, \quad Z \leq Q \tag{47}$$

$$\text{or (ii)} \quad R < \frac{\alpha + \gamma K_1}{a-\beta}, \quad Y \leq Q \tag{48}$$

$$\text{or (iii)} \quad R = \frac{\alpha + \gamma K_1}{a-\beta}, \quad Y = Z \leq Q. \tag{49}$$

If these conditions are not satisfied, then the above solution is not optimal. If $Z > Q$, then the initial catch $C(0)$ required by this policy would be more that what is permitted. We will then have to start by taking $C(0) = Q$. So long as the catch $C(t) = Q$, we have

$$\frac{dX}{dt} = aX - Q \tag{50}$$

(i) If $aR > Q$, then initially $\frac{dX}{dt} > 0$ and the population continues to increase and

$$aX + \left(X - \frac{\alpha + \gamma K_1}{a-\beta}\right)(a - \lambda - \beta) \tag{51}$$

always remains greater than Q and there is no solution to the optimal policy obtained above. In this case after some time

$$\alpha + \beta X + \gamma E = Q \tag{52}$$

cannot hold because this would require E to be negative. The model has to be modified.

(ii) If $aR = Q$ and we fish at a constant rate Q, then $X(t)$ and $E(t)$ remain constant.

(iii) If $aR < Q$, the population goes on decreasing till it becomes zero at time t_1, given by

$$Q = (Q - aR)\, e^{at_1} \tag{53}$$

2.2.8 VARIATION OF E(O) AND OF EQUILIBRIUM VALUES OF E AND X WITH THE PARAMETERS

(a) E(O) is given by

$$E(0) = \frac{p\gamma}{2b} - \frac{k(\lambda - a + \beta)}{2b(a-\beta)} + \frac{R}{\gamma}(2a - 2\beta - \lambda)$$

$$- \frac{p\beta\gamma}{2b(a-\beta)} - \frac{\gamma(2a - 2\beta - \lambda)}{2b(a-\beta)}\left(p + \frac{2b\alpha}{\gamma^2}\right).$$

Since $a > \beta$, $2a - 2\beta - \lambda > 0$, we deduce that

(i) E(O) increases with R i.e. initial optimal effort increases as the initial population increases.

(ii) E(O) increases with k if $k > 0$.

(iii) E(O) decreases with p i.e. the initial optimal effort decreases with the increase in the selling price of the fish.

(iv) E(O) increases with b.

(v) E(O) decreases with α.

(vi) E(O) is independent of ℓ .

(b) The Limiting value E_∞ of $E(t)$ as $t \to \infty$ is given by

$$K_1 = \frac{p\gamma - k}{2b} + \frac{p\gamma\beta}{2b(a-\lambda-\beta)}$$

Since $a-\beta-\lambda > 0$, we deduce that

(i) E_∞ decreases with b.

(ii) E_∞ increases with p.

(iii) E_∞ decreases with k if $k > 0$.

(iv) E_∞ increases with γ.

(v) E_∞ decreases with a.

(vi) E_∞ increases with β.

(vii) E_∞ increases with λ.

(c) The Limiting population X_∞ as $t \to \infty$ is given by

$$X_\infty = \frac{\alpha + \gamma K_1}{a-\beta} = \frac{1}{a-\beta}\left[\alpha + \frac{p\gamma^2}{2b} - \frac{k\gamma}{2b} + \frac{p\gamma^2\beta}{2b(a-\beta-\lambda)}\right]$$

Since $a-\beta > 0$, $a-\beta-\lambda > 0$, we have

(i) X_∞ increases with α.

(ii) X_∞ increases with p.

(iii) X_∞ decreases with k if $k > 0$.

(iv) X_∞ increases with γ.

(v) X_∞ decreases with a.

(vi) X_∞ increases with β.

2.2.9 EXPLOITATION OVER A FINITE HORIZON OF TIME

Suppose we want to exploit the fishery over a finite horizon of time and we want to stop just when the fishery ceases

to be profitable i.e. we shall stop when

$$p[\alpha+\beta X(T) + \gamma E(T)] - [bE^2(T) + kE(T) + \ell] = 0 \qquad (54)$$

Using equations (19) to (22)

$$p\{\alpha + \beta \frac{\alpha + \gamma K_1}{(a-\beta)} + \beta \frac{\gamma^2 A_1}{2b(\lambda-2a+2\beta)} e^{(\lambda-a+\beta)T} + B A_2 e^{(a-\beta)T}$$

$$+ \gamma K_1 - \frac{A_1 \gamma^2}{2b} e^{(\lambda-a+\beta)T}\}$$

$$- b\{K_1 - \frac{A_1\gamma}{2b} e^{(\lambda-a+\beta)T}\}^2 - k\{K_1 - \frac{A_1\gamma}{2b} e^{(\lambda-a+\beta)T}\}$$

$$- \ell = 0 \qquad (55)$$

Also since $X(0) = R$, we get

$$\frac{\alpha + \gamma K_1}{a-\beta} + \frac{\gamma^2 A_1}{2b(\lambda-2a+2\beta)} + A_2 = R \qquad (56)$$

From (55) and (56) we can determine the two arbitrary constants A_1 and A_2. In fact we may get two sets of values of A_1 and A_2. Equations (19) and (22) then determine $E(t)$ and $X(t)$.

We have also to ensure that

$$p\{\alpha + \beta X(0) + \gamma E(0)\} - \{b E^2(0) + k E(0) + \ell\} > 0, \quad (57)$$

otherwise, we do not exploit the fishery and allow the fish population to grow to a larger value of R.

We may note that if we have a finite horizon of time, then the restrictions (24) and (25) are not necessary.

Even if we start with the optimal exploitation over an infinite horizon of time, and use (27) and (28) for $X(t)$ and $E(t)$ it would be profitable to stop when (54) is satisfied since any further exploitation would lead to negative profits.

2.2.10 CONCLUDING REMARKS

(i) If instead of (3) we use

$$\frac{dX}{dt} = aX - dX^2 - \beta X - \alpha - \gamma E \qquad (58)$$

then equation (16) of Sancho and Mitchell [1975] is modified to

$$\lambda f = p(\alpha+\beta R) + \frac{1}{4b}\left(p - \frac{\partial f}{\partial R}\right)^2 \gamma^2 + \{(a-\beta)R - dR^2 - \alpha\} \frac{\partial f}{\partial R} \qquad (59)$$

If $d = 0$, it admits of a simple solution

$$f(R) = AR^2 + BR + K$$

but if $d \neq 0$, such a simple solution is not possible. If we try a third degree polynomial, we get five equations to determine four constants and similarly for nth degree polynomial, we get one more equations than the number of constants. The system is over-determined and a unique solution does not exist.

(ii) When we use $q(E)$ to be cubic function of E, we get mathematically difficult problems by all three methods, Kapur and Saleem [1977].

(iii) When we use (58) with maximum principle or calculus of variations we get non-linear differential equation for $X(t)$ which we hope to be able to solve numerically, Kapur and Saleem [1977].

(iv) From dynamic programming solution, we can also find both $X(t)$ and $E(t)$ using (11), (8) and (31) (which gives the correct form for (9)), we get (33) as

$$E(0) = K_1 - \frac{(\lambda - 2a + 2\beta)}{\gamma} \left(R - \frac{\alpha + \gamma K_1}{a-\beta}\right) \tag{60}$$

Since $E(t)$ is the same function of $X(t)$ as $E(0)$ is of $X(0)$, we get

$$E(t) = K_1 - \frac{(\lambda - 2a + 2\beta)}{\gamma} \left(X(t) - \frac{\alpha + \gamma K_1}{a-\beta}\right) \tag{61}$$

Substituting in (3) and integrating the resulting differential equation and using $X(0) = R$, we get the expression for $X(t)$ given by (27), substituting this in (35) we get the expression for $E(t)$ given by (28).

2.3 MULTISPECIES FISHERIES

The optimality principle of dynamic programming and the maximum principle have been used to maximize the present-value of profits over all time in a fishery exploitation when the fishery has a number of different but non-interacting species.

Sancho and Mitchell [1975] maximized the present-value of total profits over time given by equation [5, Section 2.2]. Subject to [3, Section 2.2] when there was only one species involved.

However the area of the pond or a river or a sea may contain a number of different types of fish commonding different

prices and all of them may be cought. We assume that when an effort E is made, the catch of the ith species ($i=1,2,3,4,\ldots,n$) is given by

$$C_i(t) = \alpha_i + \beta_{i1}X_1 + \beta_{i2}X_2 + \ldots + \beta_{in}X_n + \gamma_i E \quad (1)$$

and the selling price of the ith species is p_i so that present value P of the profits is given by

$$P = \int_0^{\infty} e^{-\lambda t} \left[\sum_{i=1}^{n} \rho_i(\alpha_i + \sum_{j=1}^{n} \beta_{ij}X_j + \gamma_i E) - bE^2 - kE - \ell \right] dt \quad (2)$$

We have to maximize P subject to

$$\frac{dX_i}{dt} = a_i X_i - (\alpha_i + \sum_{j=1}^{n} \beta_{ij} X_j + \gamma_i E) \; (i = 1,2,3,\ldots,n) \quad (3)$$

2.3.1 USE OF OPTIMALITY PRINCIPLE

Let $f(R_1,R_2,\ldots,R_n)$ be the maximum value of P, where

$$X_i(0) = R_i, \quad (4)$$

so that R_1, R_2, $R_3,\ldots$, R_n are the initial populations of n species respectively.

The optimality principle then gives

$$f(R_1,R_2,\ldots,R_n) = \max_E \Big[\Delta \{ \sum_{i=1}^{n} p_i(\alpha_i + \sum_{j=1}^{n} \beta_{ij}R_j + \gamma_i E) - bE^2 - kE - \ell \} + e^{-\lambda\Delta} f\{ R_1 + \Delta(a_1R_1 - \alpha_1 - \sum_{j=1}^{n} \beta_{1j}R_j - \gamma_1 E), \ldots, R_n + \Delta(a_nR_n - \alpha_n - \sum_{j=1}^{n} \beta_{nj} R_j - \gamma_n E) \} \Big], \quad (5)$$

where E is the initial effort and we have made use of (3) to find the increments in populations.

Using Taylor's theorem, expanding in powers of Δ, cancelling same terms in both sides of (5) and taking the limit as $\Delta \to 0$, we get

$$f(R_1,R_2,\ldots,R_n) = \max_E \Big[\Delta \{ \sum_{i=1}^{n} p_i(\alpha_i + \sum_{j=1}^{n} \beta_{ij} R_j + \gamma_i E) - bE^2 - kE - \ell \} + (1 - \lambda\Delta + \frac{\lambda^2 \Delta^2}{\underline{|2}} - \ldots) \{ f(R_1,R_2,\ldots,R_n) + \Delta \sum_{i=1}^{n} ((a_i R_i - \alpha_i - \sum_{j=1}^{n} \beta_{ij} R_j) \frac{\partial f}{\partial R_i}) + \ldots \} \Big] \tag{6}$$

$$\lambda f(R_1,R_2,\ldots,R_n) = \max_E \Big[\sum_{i=1}^{n} p_i(\alpha_i + \sum_{j=1}^{n} \beta_{ij} R_j + \gamma_i E) - bE^2 - kE - \ell + \sum_{i=1}^{n} \{ a_i R_i - (\alpha_i + \sum_{j=1}^{n} \beta_{ij} R_j + \gamma_i E) \} \frac{\partial f}{\partial R_i} \Big] \tag{7}$$

This gives the following two equations to determine E and $f(R_1,R_2,\ldots,R_n)$

$$\sum_{i=1}^{n} (p_i \gamma_i - \gamma_i \frac{\partial f}{\partial R_i}) - 2bE - k = 0 \tag{8}$$

$$\lambda f(R_1,R_2,\ldots,R_n) = \sum_{i=1}^{n} p_i(\alpha_i + \sum_{j=1}^{n} \beta_{ij} R_j) - \ell + \sum_{i=1}^{n} (a_i R_i - \alpha_i - \sum_{j=1}^{n} \beta_{ij} R_j) \frac{\partial f}{\partial R_i} + \frac{1}{4b} \{ \sum_{i=1}^{n} \gamma_i (p_i - \frac{\partial f}{\partial R_i}) - k \}^2 \tag{9}$$

The equation (8) comes because we also maximize $f(R_1,\ldots,R_n)$ as a function of E. Equation (9) is obtained after substituting the value of E from (8) into (7).

2.3.2 SOLUTION OF THE EQUATIONS

Using the quadratic nature of the problem, we assume a solution of (9) of the form

$$f(R_1,R_2,\ldots,R_n) = \sum_{j=1}^{n}\sum_{i=1}^{n} A_{ij}\, R_i\, R_j + 2\sum_{i=1}^{n} B_i\, R_i + C, \quad (10)$$

where $A_{ij} = A_{ji}$.

Substituting from (10) in (9) and equating the coefficients of $R_1^2, R_2^2,\ldots,R_n^2$; $R_1, R_2,\ldots,R_{n-1}\, R_n$; $R_1, R_2,\ldots,R_n$ and the constant term, we get $\frac{1}{2}(n^2 + 3n + 2)$ equations to determine the same number of constants A_{ij}'s, B_i's and C. Thus following Bellman [1967], a unique solution of our problem of the form (10) exists.

We can determine E from (8)

$$E(0) = \frac{1}{2b}\left[\sum_{i=1}^{n} \gamma_i \{p_i - 2(\sum_{j=1}^{n} A_{ij}\, R_j + B_i)\} - k\right] \qquad (11)$$

Equation (10) gives the maximum value of P and equation (11) determines the initial effort required.

2.3.3 FURTHER INTEGRATION OF EQUATIONS

The optimal effort at time t depends on the population at time t in the same way as the optimal effort initially depends on the initial population so that

$$E(t) = \frac{1}{2b} \{ \sum_{i=1}^{n} \gamma_i(p_i - 2 \sum_{j=1}^{n} A_{ij} X_j(t) - 2B_i) - k \} \quad (12)$$

Substituting from (12) in (3), we get

$$\frac{dX_i}{dt} = a_i X_i - \alpha_i - \sum_{j=1}^{n} \beta_{ij} X_j - \frac{\gamma_i}{2b} \sum_{i=1}^{n} \gamma_i p_i + \frac{\gamma_i}{b} \sum_{i=1}^{n} \sum_{j=1}^{n} A_{ij} \gamma_i X_j(t)$$

$$+ \frac{\gamma_i}{b} \sum_{i=1}^{n} B_i \gamma_i + \frac{k\gamma_i}{2b} \quad (13)$$

We get a set of n simultaneous differential equations which can be easily solved subject to

$$X_1(0) = R_1,\ X_2(0) = R_2, \ldots, X_n(0) = R_n \quad (14)$$

to get $X_1(t)$, $X_2(t), \ldots, X_n(t)$. Substituting in (12) we get $E(t)$ as a function of t. Substituting these functions in (2) we can get back $f(R_1, R_2, \ldots, R_n)$ as the maximum value of P.

The solutions of the equations are of the form

$$X_i(t) = \sum_{j=1}^{n} C_{ij}\, e^{\lambda_j t} + k_i \quad (15)$$

$$E(t) = \sum_{j=1}^{n} f_j\, e^{\lambda_j t} + f,$$

where $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n$ are the roots of the equation

$$|D - \lambda I| = 0, \quad (16)$$

where D is an n x n matrix whose i-jth element is given by

$$a_i\, \delta_{ij} - \beta_{ij} + \frac{\gamma_i}{b} \sum_{=1}^{n} A_{\ell j}\, \gamma_\ell \quad (17)$$

For $X(t)$ and $E(t)$ to be finite as $t \to \infty$, we would require that all the roots of this equation should have negative real parts. The condition for this can be written by using Routh-Hurwitz criterion or Lienard-Chipart test vide Korn and Korn [1968].

2.3.4 USE OF MAXIMUM PRINCIPLE

From (2) and (3) we form the Hamiltonian

$$H = e^{-\lambda t} \{ \sum_{i=1}^{n} p_i(\alpha_i + \sum_{j=1}^{n} \beta_{ij} X_j + \gamma_i E) - bE^2 - kE - \ell \}$$

$$+ \sum_{i=1}^{n} \psi_i(t) \; \{ a_i X_i - (\alpha_i + \sum_{j=1}^{n} \beta_{ij} X_j + \gamma_i E) \} \tag{18}$$

The auxiliary equations are

$$\frac{\partial H}{\partial X_i} = - \frac{d\psi_i}{dt} \quad (i = 1,2,3,\ldots,n) \tag{19}$$

These give the n linear differential equations

$$e^{-\lambda t} \sum_{j=1}^{n} p_j \beta_{ji} + a_i \psi_i(t) - \sum_{j=1}^{n} \beta_{ji} \psi_j(t) = - \frac{d\psi_i}{dt} , \tag{20}$$

of which the solutions are given by

$$\psi_i(t) = \sum_{j=1}^{n} A_{ij} e^{\mu_j t} + K_i e^{-\lambda t}, \; (i=1,2,3,\ldots,n)$$

where $\mu_1, \mu_2, \ldots, \mu_n$ are the roots of

$$\begin{vmatrix} \mu+a_1-\beta_{11} & -\beta_{12} & -\beta_{13} & \cdots & -\beta_{1n} \\ -\beta_{21} & \mu+a_2-\beta_{22} & -\beta_{23} & \cdots & -\beta_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ -\beta_{n1} & -\beta_{n2} & -\beta_{n3} & \cdots & \mu+a_n-\beta_{nn} \end{vmatrix} = 0 \quad (21)$$

The total number of arbitrary constants in (21) would be n. Now H has to be maximized as a function of E. This gives

$$e^{-\lambda t}\left(\sum_{i=1}^{n} p_i \gamma_i - 2bE - k\right) - \sum_{i=1}^{n} \gamma_i \psi_i(t) = 0, \qquad (22)$$

so that

$$E = \frac{1}{2b}\left(\sum_{i=1}^{n} p_i \gamma_i - k - e^{\lambda t} \sum_{i=1}^{n} \sum_{j=1}^{n} \gamma_i A_{ij} e^{\mu_j t} - \sum_{i=1}^{n} k_i \gamma_i\right)$$

(23)

Substituting in (3) we get n linear differential equations to solve for $X_1(t)$, $X_2(t), \ldots, X_n(t)$. The solution would contain in all 2n arbitrary constants, for determinations of which we have the conditions

$$X_i(0) = R_i \qquad (24)$$

and that $X_i(t)$ should remain finite as $t \to \infty$.

We solve the equations in the special case when

$$\beta_{ii} = \beta_i, \; \beta_{ij} = 0, \quad \text{when } j \neq i \qquad (25)$$

i.e. when the catch of the ith species per unit time depends on only its own population and the effort E. In this case (20) becomes

$$\frac{d\psi_i}{dt} = -(a_i - \beta_i)\ \psi_i - p_i\ \beta_i\ e^{-\lambda t} \tag{26}$$

giving

$$\psi_i(t) = -\ \frac{p_i\ \beta_i\ e^{-\lambda t}}{a_i - \beta_i - \lambda} + A_i\ e^{-(a_i-\beta_i)t} \tag{27}$$

Substituting in (22)

$$2bE = \sum_{i=1}^{n} p_i\ \gamma_i - k - e^{\lambda t} \sum_{i=1}^{n} \gamma_i\ A_i\ e^{-(a_i-\beta_i)t}$$

$$+ \sum_{i=1}^{n} \frac{p_i\ \beta_i\ \gamma_i}{a_i - \beta_i - \lambda} \tag{28}$$

Substituting in (3)

$$\frac{dX_i}{dt} = (a_i-\beta_i)X_i - \alpha_i - \frac{\gamma_i}{2b}\ \{ \sum_{i=1}^{n} p_i\gamma_i + \sum_{i=1}^{n} \frac{p_i\ \beta_i\ \gamma_i}{a_i - \beta_i - \lambda} - k$$

$$- \sum_{i=1}^{n} \gamma_i\ A_i\ e^{(\lambda - a_i + \beta_i)t} \} \tag{29}$$

Integrating

$$X_i(t) = \{ \alpha_i + \frac{\gamma_i}{2b} \sum_{i=1}^{n} \frac{p_i\ \gamma_i(a_i - \lambda)}{a_i - \beta_i - \lambda} - \frac{k\ \gamma_i}{2b} \}\ \frac{1}{a_i - \beta_i}$$

$$+ \frac{\gamma_i}{2b} \sum_{j=1}^{n} \gamma_j A_j\ \frac{e^{(\lambda - a_j + \beta_j)t}}{\lambda - a_j + \beta_j - a_i + \beta_i} + M_i\ e^{(a_i-\beta_i)t}, \tag{30}$$

where A_j's and M_i's are arbitrary constants.

Since we are assuming that $a_i > \beta_i$ and $\lambda < (a_i - \beta_i)$ and we want $X_i(t)$ to remain finite as $t \to \infty$, we get $M_i = 0$.

Also

$$R_i = \{ \alpha_i + \frac{\gamma_i}{2b} \sum_{i=1}^{n} \frac{p_i \ \gamma_i(a_i - \lambda)}{a_i - \beta_i - \lambda} - \frac{k \ \gamma_i}{2b} \} \frac{1}{a_i - \beta_i}$$

$$+ \frac{\gamma_i}{2b} \sum_{j=1}^{n} \frac{\gamma_j \ A_j}{\lambda - a_j + \beta_j - a_i + \beta_i} \tag{31}$$

These are n equations to determine the n constants $A_1, A_2, \ldots, A_n$. It is obvious that each A_i is a linear function of $R_1, R_2, \ldots, R_n$. We then get

$$X_i(t) = D_i + \frac{\gamma_i}{2b} \sum_{j=1}^{n} \frac{\gamma_j A_j}{\lambda - a_j + \beta_j - a_i + \beta_i} e^{(\lambda - a_j + \beta_j)t} \tag{32}$$

where

$$D_i = \{ \alpha_i + \frac{\gamma_i}{2b} \sum_{i=1}^{n} \frac{p_i \ \gamma_i(a_i - \lambda)}{a_i - \beta_i - \lambda} - \frac{k \ \gamma_i}{2b} \} \frac{1}{a_i - \beta_i}$$

$$2bE = 2b \ E_o - \sum_{i=1}^{n} \gamma_i \ A_i \ e^{(\lambda - a_i + \beta_i)t} \tag{33}$$

The maximum profit is given by

$$\int_o^\infty \bar{e}^{\lambda t} [\sum_{i=1}^{n} p_i(\alpha_i + \beta_i X_i + \gamma_i E) - bE^2 - kE - \ell] \ dt \tag{34}$$

Substituting from (32) and (33) in (34) and integrating, we get the maximum profit as a quadratic function of $A_1, A_2, \ldots, A_n$, but since $A_1, A_2, \ldots, A_n$ are themselves linear functions of $R_1, R_2, \ldots, R_n$, we get the maximum profit as a quadratic function of $R_1, R_2, \ldots, R_n$.

As $t \to \infty$

$$X_i(t) \to \left[\alpha_i + \frac{\gamma_i}{2b} \sum_{i=1}^{n} \frac{p_i\, \gamma_i (a_i - \lambda)}{a_i - \beta_i - \lambda} - \frac{k\, \gamma_i}{2b}\right] \frac{1}{a_i - \beta_i} \tag{35}$$

$$E(t) \to \frac{1}{2b}\left[\sum_{i=1}^{n} \gamma_i \frac{p_i\,(a_i - \lambda)}{a_i - \beta_i - \lambda} - k\right] \tag{36}$$

For $n = 1$, these results tally with the results obtained earlier. We note that

(i) The limiting values of $X_i(t)$ and $E(t)$ are independent of $R_1, R_2, \ldots, R_n$.

(ii) When $k = 0$, the ultimate effort is the sum of n independent components due to individual species. However when $k > 0$, the ultimate effort required is more than the sum of the n components due to n species.

(iii) The limiting population of the ith species is more than what it would be in the absence of other species.

(iv) The effort required initially as well as at any time depends on the initial populations and the parameters of all the species.

(v) We find $X_i(\infty) = \frac{1}{a_i - \beta_i}(\alpha_i + \gamma_i\, E(\infty))$,

where $X_i(\infty)$ and $E(\infty)$ denote the limits of $X_i(t)$ and $E(t)$ as $t \to \infty$. This is expected since ultimately $\frac{dX_i}{dt}$ should vanish in

$$\frac{dX_i}{dt} = (a_i - \beta_i) X_i(t) - \alpha_i - \gamma_i\, E(t).$$

Thus ultimately the population of each species is more than it would be in the absence of other species and the ultimate effort required is more than it would be for any individual species. As $t \to \infty$, the catch per unit time as well as the growth of fish of each species would be greater than in the absence of other species, but the rate of growth of each species would be greater than the catching rate of that species so that its ultimate population would be greater.

2.3.5 PARTICULAR CASE OF TWO SPECIES

We have already solved the problem for n species. For two species, we put $n = 2$ in all the formulae obtained earlier in 2.3.1 with an assumption that the catch per unit time of the ith species depends only on its population so that the equation (1) becomes

$$C_i(t) = \alpha_i + \beta_i X_i + \gamma_i E \quad (i = 1,2,) \tag{37}$$

Assuming

$$f(R_1,R_2) = \sum_{j=1}^{2} \sum_{i=1}^{2} A_{ij} R_i R_j + 2 \sum_{i=1}^{2} B_i R_i + D, \tag{38}$$

where $A_{ij} = A_{ji}$. Substituting from (38) in (9) with all the changes taking into account including $k = 0$ and $\ell = 0$ to make the calculations simpler and equating the coefficients of R_1^2, R_2^2, R_1R_2, R_1, R_2 and the constant term on both sides, we get the following six equations

$$\lambda A_{11} = 2(a_1-\beta_1) A_{11} + \frac{1}{b} (A_{11} \gamma_1 + A_{12} \gamma_2)^2 \tag{39}$$

$$\lambda A_{22} = 2(a_2-\beta_2)\, A_{22} + \frac{1}{b}\,(A_{12}\,\gamma_1 + A_{22}\,\gamma_2)^2 \tag{40}$$

$$2\lambda A_{12}=2(a_1-\beta_1)A_{12}+2(a_2-\beta_2)A_{12}+ \frac{2}{b}\,(A_{11}\gamma_1+A_{12}\gamma_2)(A_{12}\,\gamma_1+A_{22}\gamma_2) \tag{41}$$

$$2\lambda B_1=p_1\beta_1+2(a_1-\beta_1)B_1-2A_{11}\alpha_1-2A_{12}\alpha_2- \frac{1}{b}\,(\gamma_1p_1+\gamma_2p_2-2B_1\gamma_1-2B_2\gamma_2)(A_{11}\gamma_1 + A_{12}\gamma_2) \tag{42}$$

$$2\lambda\, B_2 = p_2\beta_2 + 2(a_2-\beta_2)B_2 - 2\,\alpha_1\, A_{12} - 2\alpha_2\, A_{22} - \frac{1}{b}\,(\gamma_1p_1+\gamma_2p_2-2B_1\gamma_1-2B_2\gamma_2)(A_{12}\gamma_1+A_{22}\gamma_2) \tag{43}$$

$$\lambda D = p_1\alpha_1 + p_2\alpha_2 - 2\alpha_1B_1 - 2\alpha_2B_2 + \frac{1}{4b}\,(p_1\gamma_1+p_2\gamma_2 - 2B_1\,\gamma_1-2B_2\gamma_2)^2 \tag{44}$$

We write equations (39), (40) and (41) in the form

$$k_1\, A = (\gamma_1\, A + \gamma_2\, C)^2 \tag{45}$$

$$k_2\, B = (\gamma_1\, C + \gamma_2\, B)^2 \tag{46}$$

$$k_3\, C = (\gamma_1\, A + \gamma_2\, C)\,(\gamma_1\, C + \gamma_2\, B), \tag{47}$$

where $A = A_{11}$ $\quad k_1 = b(\lambda - 2a_1 + 2\beta_1)$

$B = A_{22}$ $\quad k_2 = b(\lambda - 2a_2 + 2\beta_2)$

$C = A_{12}$ $\quad k_3 = b(\lambda - a_1 + \beta_1 - a_2 + \beta_2)$

From (45), (46) and (47), we find

$$k_1\, k_2\, AB = k_3^2\, C^2 \tag{48}$$

Substituting the value of A from (48) in (45), we get the following two equation in B and C to be solved

$$\gamma_1^2 k_3^4 C^2 + 2\gamma_1\gamma_2 k_1 k_2 k_3^2 BC + \gamma_2^2 k_1^2 k_2^2 B^2 - k_1^2 k_3^2 k_2 B = 0 \qquad (49)$$

$$\gamma_1^2 C^2 + 2\gamma_1\gamma_2 BC + \gamma_2^2 B^2 - k_2 B = 0 \qquad (50)$$

Eliminating C, we obtain

$$Q^2 = PR, \qquad (51)$$

where $P = 2\gamma_2 k_1 k_2 k_3^2 B(\gamma_2^2 B - k_2) - 2\gamma_2 k_1^2 k_2 B(\gamma_2^2 k_2 B - k_3^2)$

$Q = \gamma_1 k_1^2 k_2(\gamma_2^2 k_2 B - k_3^2) - \gamma_1 k_3^4 (\gamma_2^2 B - k_2)$

$R = 2\gamma_1^2 \gamma_2 k_3^2 (k_3^2 - k_1 k_2)$.

We get a quadratic equation in B to be solved. This gives the value of A_{22}. Substituting the value of B in (46) we obtain C i.e. A_{12} and again substituting the value of C in (45) we have A_{11}. Similarly, from equations (42), (43) and (44) we obtain B_1, B_2 and D. Thus knowing all the constants we can get the maximum profit $f(R_1,R_2)$.

2.4 LIMITING OPTIMAL CONTROL IN FISHERIES

In this section it is shown that limiting optimal solutions may be possible for an ecological system which may not, otherwise, be tractable. We consider here the general system. As a special case, the system considered by Clark [1976] is solved completely for optimal equilibrium solutions. The case for interacting species is also considered.

The problem of optimal exploitation of fisheries bristles with mathematical difficulties. In a recent survey Clark [1976] states

" Our problem therefore possesses an equilibrium solution satisfying the necessary conditions for the maximum principle. However economic interpretation of this solution does not seem obvious. However, it appears to be extremely difficult to determine the optimal approach path which must consist of some combinations of bang-bang controls and non-equilibrium singular controls. We shall not make further attempts to obtain complete solution to the problem. The difficulty in this problem seems to be associated with the non-controllability of the system given by

$$\frac{dx}{dt} = rx\ (1 - \frac{x}{K}) - q_1\ Ex$$

$$\frac{dy}{dt} = sy\ (1 - \frac{y}{L}) - q_2\ Ey, \tag{1}$$

in the sense that $\dot{x}$ and $\dot{y}$ cannot be controlled independently".

However in the previous section 2.3 we have used both dynamic programming and maximum principle to obtain the optimal solution for the system given by

$$\frac{dx}{dt} = a_1 x - (\alpha_1 + \beta_1 x + \gamma_1 E)$$

$$\frac{dy}{dt} = a_2 y - (\alpha_2 + \beta_2 y + \gamma_2 E) \tag{2}$$

inspite of the fact that here also $\dot{x}$ and $\dot{y}$ cannot be controlled independently. The difficulty may therefore really be in the

non-linear nature of the functions concerned, though this may be complicated by the difficulty of independent control.

2.4.1 BASIC EQUATIONS

For discussing in a general context, we consider the system given by

$$\frac{dx}{dt} = F_1(x) - EG_1(x) - H_1(x)$$

$$\frac{dy}{dt} = F_2(y) - EG_2(y) - H_2(y), \tag{3}$$

which includes both (1) and (2) as special cases. Here $x(t)$ and $y(t)$ are the biomasses of the two species of fish, E is the effort, $F_1(x)$ and $F_2(y)$ are the natural rates of growth of fish and $EG_1(x) + H_1(x)$ and $EG_2(x) + H_2(y)$ are the catches of the two species of fish per unit time.

If p_1 and p_2 are the two prices of the two types of fish per unit biomass, λ is the instantaneous discount rate and the cost of an effort E is $bE^2 + kE + \ell$, then the present value of the profit is given by

$$P = \int_0^\infty e^{-\lambda t} \{p_1(EG_1(x) + H_1(x)) + p_2 (EG_2(y) + H_2(y)) - (bE^2 + kE + \ell)\} \, dt \tag{4}$$

We have to choose E(t) so as to maximize (4) subject to (3).

We form the Hamiltonian [Pontriagin, 1963]

$$H = \bar{e}^{\lambda t} \{p_1(EG_1(x) + H_1(x)) + p_2(EG_2(y) + H_2(y))$$
$$- (bE^2 + kE + \ell)\} + \psi_1(t) \{F_1(x) - EG_1(x) - H_1(x)\}$$
$$+ \psi_2(t) \{F_2(y) - EG_2(y) - H_2(y)\} \qquad (5)$$

Using the auxiliary equations

$$\frac{\partial H}{\partial x} = - \frac{d\psi_1}{dt} , \quad \frac{\partial H}{\partial y} = - \frac{d\psi_2}{dt} \qquad (6)$$

and maximizing H as a function of E, we get

$$\bar{e}^{\lambda t} \{p_1 EG_1'(x) + p_1 H_1'(x)\} + \psi_1(t) \{F_1'(x) - EG_1'(x) - H_1'(x)\}$$
$$= - \frac{d\psi_1}{dt} \qquad (7)$$

$$\bar{e}^{\lambda t} \{p_2 EG_2'(y) + p_2 H_2'(y)\} + \psi_2(t) \{F_2'(y) - EG_2'(y) - H_2'(y)\}$$
$$= - \frac{d\psi_2}{dt}$$

and

$$\bar{e}^{\lambda t} \{p_1 G_1(x) + p_2 G_2(y) - 2bE - k\} - \psi_1(t) G_1(x) - \psi_2(t) G_2(y)$$
$$= 0 \qquad (8)$$

From (3), (7) and (8), we get a system of five equations for determining the five functions $x(t)$, $y(t)$, $E(t)$, $\psi_1(t)$ and $\psi_2(t)$.

We can solve for E from (8) and substitute in (3) and (7) to get four differential equations of the first order to solve for $x(t)$, $y(t)$, $\psi_1(t)$ and $\psi_2(t)$. These can be integrated

numerically if $x(0)$, $y(0)$, $\psi_1(0)$ and $\psi_2(0)$ are known. Now $x(0) = R_1$, $y(0) = R_2$, where R_1, R_2 are the initial populations. $\psi_1(0)$ and $\psi_2(0)$ are not known. Instead we have the conditions that $x(t)$, $y(t)$, $\psi_1(t)$, $\psi_2(t)$ and $E(t)$ should tend to finite limits as t tends to infinity. Alternatively if we want to exploit over a finite horizon of T time, we can use the conditions $X(T) = 0$, $y(T) = 0$.

2.4.2 PARTICULAR CASE

Equations (3) reduce to equation (2) if

$$\begin{aligned} F_1(x) &= a_1 x \quad , \; F_2(y) = a_2 y \\ G_1(x) &= \gamma_1 \quad , \; G_2(y) = \gamma_2 \qquad\qquad (9) \\ H_1(x) &= \alpha_1+\beta_1 x, \; H_2(y) = \alpha_2+\beta_2 y \end{aligned}$$

(See in Section 2.3,2.3.4).

2.4.3 OTHER PARTICULAR CASES

It is easily seen that equation (7) will be independent of x and y if

$$\begin{aligned} G_1'(x) &= 0 \quad , \quad G_2'(y) = 0 \\ H_1'(x) &= \text{constant}, \; H_2'(y) = \text{constant} \qquad\qquad (10) \\ F_1'(x) &= \text{constant}, \; F_2'(y) = \text{constant} \end{aligned}$$

which gives the case discussed in 2.4.2. It is only in this case that we can solve for $\psi_1(t)$ and $\psi_2(t)$ separately and then get two independent equations to solve for $x(t)$ and $y(t)$.

2.4.4 (a) OPTIMAL EQUILIBRIUM SOLUTIONS

We investigate the possibility of

$$x(t) = x_o \ , \ y(t) = y_o, \ E(t) = E_o,$$
$$\psi_1(t) = C_1 \bar{e}^{\lambda t}, \ \psi_2(t) = C_2 \ \bar{e}^{\lambda t} \qquad (11)$$

as solutions of five equations (3), (7) and (8). We get

$$F_1(x_o) - E_o \ G_1(x_o) - H_1(x_o) = 0$$

$$F_2(y_o) - E_o \ G_2(y_o) - H_2(y_o) = 0$$

$$p_1 E_o G_1'(x_o) + p_1 H_1'(x_o) + C_1 [F_1'(x_o) - E_o G_1'(x_o) - H_1'(x_o)] = C_1 \lambda \qquad (12)$$

$$p_2 E_o G_2'(y_o) + p_2 H_2'(y_o) + C_2 [F_2'(y_o) - E_o G_2'(y_o) - H_2'(y_o)] = C_2 \lambda$$

$$p_1 G_1(x_o) + p_2 G_2(y_o) - 2bE_o - k - C_1 G_1(x_o) - C_2 G_2(y_o) = 0$$

In general these will determine the five constants x_o, y_o, E_o, C_1 and C_2 but we further require $x_o \geq 0$, $y_o \geq 0$, $E_o \geq 0$ and both x_o, y_o should not be zero.

For the particular case of (9), we get

$$a_1 x_o - E_o \gamma_1 - (\alpha_1 + \beta_1 x_o) = 0$$

$$a_2 y_o - E_o \gamma_2 - (\alpha_2 + \beta_2 y_o) = 0$$

$$p_1 \beta_1 + C_1(a_1 - \beta_1) = C_1 \lambda \qquad (13)$$

$$p_2 \beta_2 + C_2(a_2 - \beta_2) = C_2 \lambda$$

$$p_1 \gamma_1 + p_2 \gamma_2 - 2bE_o - k - C_1 \gamma_1 - C_2 \gamma_2 = 0$$

Solving, we get

$$x_o = \frac{1}{a_1-\beta_1}\left[\alpha_1 + \frac{\gamma_1}{2b}\left\{\frac{p_1\gamma_1(\lambda - a_1)}{\lambda - a_1 + \beta_1} + \frac{p_2\,\gamma_2(\lambda - a_2)}{\lambda - a_2 + \beta_2} - k\right\}\right]$$

$$y_o = \frac{1}{a_2-\beta_2}\left[\alpha_2 + \frac{\gamma_2}{2b}\left\{\frac{p_1\gamma_1(\lambda - a_1)}{\lambda - a_1 + \beta_1} + \frac{p_2\gamma_2\,(\lambda - a_2)}{\lambda - a_2 + \beta_2} - k\right\}\right]$$

$$2b\,E_o = p_1\gamma_1 + p_2\gamma_2 - k - \frac{p_1\,\beta_1\,\gamma_1}{\lambda - a_1 + \beta_1} - \frac{p_2\,\beta_2\,\gamma_2}{\lambda - a_2 + \beta_2} \qquad (14)$$

$$C_1 = \frac{p_1\beta_1}{\lambda - a_1 + \beta_1}\ , \quad C_2 = \frac{p_2\beta_2}{\lambda - a_2 + \beta_2}\ .$$

It is easily seen that x_o, y_o, E_o are the limits of $x(t)$, $y(t)$ and $E(t)$ as $t \to \infty$ and C_1, C_2 are the limits of $\psi_1\, e^{\lambda t}$, $\psi_2\, e^{\lambda t}$ as $t \to \infty$.

In the general case, eliminating C_1 and C_2 from the last three equations of (12) we get

$$\begin{aligned} & p_1\, G_1(x_o) + p_2\, G_2(y_o) - 2b\, E_o - k \\ & \quad - G_1(x_o)\left[\frac{p_1\, E_o\, G_1'(x_o) + p_1\, H_1'(x_o)}{\lambda - F_o'(x_o) + E_o\, G_1'(x_o) + H_1'(x_o)}\right] \\ & \quad - G_2(y_o)\left[\frac{p_2\, E_o\, G_2'(y_o) + p_2\, H_2'(y_o)}{\lambda - F_2'(y_o) + E_o\, G_2'(y_o) + H_2'(y_o)}\right] = 0 \end{aligned} \qquad (15)$$

In general this would be a cubic in E_o. For each positive non-negative value of E_o, the first two equations of (12) would then determine X_o and y_o.

If $b = 0$, we get a quadratic in E_o.

If however $G(x) = K_1$, $G_2(y) = K_2$, (15) becomes

$$p_1K_1+p_2K_2-2bE_o-k-\frac{K_1p_1H_1'(x_o)}{\lambda-F_1'(x_o)+H_1'(x_o)}-K_2\frac{p_2H_2'(y_o)}{\lambda-F_2'(y_o)+H_2'(y_o)}=0 \tag{16}$$

which is linear equation in E_o and we get only one positive value of E_o.

If we consider the special case

$$F_1(x) = rx\,(1-\tfrac{x}{K}),\ F_2(y) = sy(1-\tfrac{y}{L}),\ G_1(x) = q_1x,$$
$$G_2(y) = q_2y,\ H_1(x) = 0,\ H_2(y) = 0,\ b = 0,\ \ell = 0, \tag{17}$$

we get from (15)

$$p_1q_1x_o+p_2q_2y_o-k-q_1x_o\frac{p_1q_1\,E_o}{\lambda-r+\frac{2rx_o}{K}+E_o\,q_1}$$
$$-\,q_2y_o\frac{p_2q_2\,E_o}{\lambda-s+\frac{2sy_o}{L}+E_o\,q_2}=0, \tag{18}$$

which gives a quadratic in E_o. This is the case considered by Clark [1976].

(b) SOLUTION

We have to solve (18) subject to

$$r(1-\frac{x_o}{K}) = q_1\,E_o,\ s(1-\frac{y_o}{L}) = q_2\,E_o \tag{19}$$

Substituting for x_o, y_o from (19) in (18), we get

$$\phi(E_o) = (p_1q_1K + p_2q_2L - k)\,(\lambda+r-q_1E_o)\,(\lambda+s-q_2E_o)$$

$$-E_o(p_1K\,\frac{q_1^2}{r} + p_2L\,\frac{q_2^2}{s})(\lambda+r-q_1E_o)(\lambda+s-q_2E_o)$$

$$-p_1q_1^2E_oK(1 - \frac{q_1E_o}{r})\,(\lambda+s-q_2E_o) - p_2q_2^2E_oL$$

$$(1 - \frac{q_2E_o}{s})\,(\lambda + r - q_1E_o) \qquad (20)$$

Now

$$\phi(-\infty) = \infty,\ \phi(0) = (p_1q_1K + p_2q_2L - k)\,(\lambda+r)\,(\lambda+s),$$

$$\phi(\infty) = -\infty \qquad (21)$$

Also

$$\phi(\frac{\lambda+r}{q_1}) = \frac{p_1\,K\lambda\,(\lambda+r)}{r}\,q_1q_2\left[\frac{\lambda+s}{q_2} - \frac{\lambda+r}{q_1}\right] \qquad (22)$$

$$\phi(\frac{\lambda+s}{q_2}) = -\frac{p_2\,L\lambda\,(\lambda+s)}{s}\,q_1q_2\left[\frac{\lambda+s}{q_2} - \frac{\lambda+r}{q_1}\right] \qquad (23)$$

Equation (21) shows that there is atleast one positive value of E_o. But the value of E_o should be less than $\frac{r}{q_1}$ and $\frac{s}{q_2}$ since from equation (19) x_o and y_o would always be positive under this condition. From (21), (22) and (23), we can be sure of only one such positive value of E_o lying between

0 and $\frac{\lambda+r}{q_1}$ when $\frac{\lambda+r}{q_1} > \frac{\lambda+s}{q_2}$ and $p_1q_1K + p_2q_2L > k$

or 0 and $\frac{\lambda+s}{q_2}$ when $\frac{\lambda+s}{q_2} > \frac{\lambda+r}{q_1}$ and $p_1q_1K + p_2q_2L > k$.

2.4.5 SOME REMARKS ON THE OPTIMAL SOLUTION

In general when we are not able to find the optimal non-equilibrium singular controls, we find the equilibrium optimal solutions x_o, y_o, E_o. If $R_1 < x_o$, $R_2 < y_o$, we do not harvest till populations rise to x_o, y_o. If either or both initial populations are more than (x_o, y_o), we do the maximum harvesting permissible by the market till x_o, y_o are reached. After (x_o, y_o) have been reached, we use the optimal effort E_o and the populations remain stable thereafter at (x_o, y_o). This approach does not give the optimal solution and in fact the solutions may be far from optimal. We can easily compare P for this case with the value of P where the initial population is (x_o, y_o).

2.4.6 CASE OF INTERACTING SPECIES

In this case, the equation (3) are replaced by

$$\begin{aligned} \frac{dx}{dt} &= F_2(x,y) - EG_1(x) - H_1(x) \\ \frac{dy}{dt} &= F_2(x,y) - EG_2(y) - H_2(y) \end{aligned} \tag{24}$$

and equations (12) are modified to

$$F_1(x_o, y_o) - E_o\, G_1(x_o) - H_1(x_o) = 0$$

$$F_2(x_o, y_o) - E_o\, G_2(y_o) - H_2(y_o) = 0$$

$$p_1\, E_o\, G_1'(x_o) + p_1\, H_1'(x_o) + C_1 \left[\left(\frac{\partial F_1}{\partial x}\right)_{x_o, y_o} - E_o G_1'(x_o) - H_1'(x_o) \right] = C_1 \lambda$$

$$p_2 E_o G_2'(y_o) + p_2 H_2'(y_o) + C_2 \left[\left(\frac{\partial F_2}{\partial y}\right)_{x_o, y_o} - E_o\ G_2'(y_o) - H_2'(y_o) \right] = C_2 \lambda$$

$$p_1 G_1(x_o) + p_2 G_2(y_o) - 2bE_o - k - C_1\ G_1(x_o) - C_2\ G_2(y_o) = 0 \tag{25}$$

Eliminating C_1, C_2 from the last three equations, we get

$$p_1 G_1(x_o) + p_2 G_2(y_o) - 2b\ E_o - k$$

$$- G_1(x_o) \frac{p_1 E_o G_1'(x_o) + p_1 H_1'(x_o)}{\lambda - \left(\frac{\partial F_1}{\partial x}\right)_{x_o, y_o} + E_o G_1'(x_o) + H_1'(x_o)}$$

$$- G_2(y_o) \frac{p_2 E_o\ G_2'(y_o) + p_2\ H_2'(y_o)}{\lambda - \left(\frac{\partial F}{\partial y}\right)_{x_o, y_o} + E_o\ G_2'(y_o) + H_2'(y_o)} = 0 \tag{26}$$

We can solve for x_o, y_o, E_o from the first two equations of (25) and (26). If $G_1(x) = \gamma_1$, $G_2(y) = \gamma_2$. (26) is a linear equation in E_o. We can solve for E_o from (26) and substitute in the first two equations of (25) to get two equations for solving for (x_o, y_o).

Thus for obtaining optimal equilibrium solutions or the limiting values of $x(t)$, $y(t)$ and $E(t)$ as $t \to \infty$, the case of interacting species presents no additional problem.

CHAPTER III

OPTIMAL EXPLOITATION OF MINES

3.1 INTRODUCTION As we have mentioned in Chapter I that studies related to exploitation of mines have not been generally concerned with the effort that we apply while extracting the material from the mine. This aspect is studied in this chapter by assuming different extraction functions where the cost function is a function of the effort applied per unit time.

The studies are aimed at providing the maximum present value of total profits over given time if the exploitation period is fixed and the maximum of all the maximum present values of total profits when the exploitation period can be varied. For the optimal solution, the amount of the material to be taken out and the effort to be applied per unit time are also plotted against time.

3.2 THE MODEL

The model used here is

$$\frac{dx}{dt} = - C(t), \qquad (1)$$

where $x(t)$ = mass of the material in the mine at time t.

and $C(t)$ = mass of the material taken out per unit time at time t.

Let $q(e)$ denote the cost of making effort e per unit time. The present value (P.V.) of total profits is given by

$$P = \int_0^T \bar{e}^{\lambda t} \left[p\ C(t) - q(e) \right] dt, \qquad (2)$$

where $\lambda > 0$ is the given instantaneous discount rate, p is the fixed price per unit of material taken out and T is the time during which the total material in the mine is taken out.

In Clark, Edwards and Friedlander [1973], q(e) was taken to be linear, but Sancho and Mitchell [1975] took it to be in the more realistic form

$$q(e) = be^2 + ke + \ell\ , \qquad (3)$$

where b, k, ℓ are parametric constants and $e(t) \geq 0$.

Thus our problem is to maximize

$$P = \int_0^T \bar{e}^{\lambda t} \left[p\ C(t) - (be^2+ke+\ell) \right] dt \qquad (4)$$

Subject to (1).

3.3 THREE EXTRACTION FUNCTIONS CONSIDERED

We consider the following three extraction functions

(a) Following Bell and Carlson [1970] and Sancho and Mitchell [1975], we take C(t) as linear in amount x(t) and effort e(t) i.e. C(t) is given by

$$C(t) = \alpha + \beta x + \gamma e,\ e(t) \geq 0 \qquad (1)$$

where α, β, γ are positive parametric constants.

(b) We take C(t) as linear in e(t) and quadratic in x(t). This may be more realistic in practice. Now C(t) is given by

$$C(t) = \alpha+\beta x + \gamma x^2 + \delta e(t), \quad e(t) \geq 0 \tag{2}$$

where α, β, γ and δ are positive parametric constants.

(c) We assume that material taken out per unit time is proportional to the product of effort and amount of the material i.e. $C(t)$ is given by

$$C(t) = \beta x(t)\, e(t), \tag{3}$$

where $x(t) \geq 0$, $e(t) \geq 0$ and β is positive parametric constant.

3.4 OPTIMIZATION FOR THE FIRST FUNCTION

For the first function, (Section 3.3, (1)), we have to maximize

$$P = \int_0^T \bar{e}^{\lambda t}\left[p(\alpha+\beta x+\gamma e) - be^2 - ke - \ell\right] dt, \tag{1}$$

subject to

$$\frac{dx}{dt} = -\alpha - \beta x - \gamma e \tag{2}$$

3.4.1 USE OF MAXIMUM PRINCIPLE

To maximize P given by (1) subject to (2), we form the Hamiltonian [Pontryagin [1963]]

$$H(t,x,e,\psi) = \bar{e}^{\lambda t}\,\{p(\alpha+\beta x+\gamma e) - be^2 - ke - \ell\} - \psi(t)\,(\alpha + \beta x + \gamma e), \tag{3}$$

where $\psi(t)$ is an auxiliary function satisfying

$$\frac{\partial H}{\partial x} = -\frac{d\psi}{dt} \tag{4}$$

We also maximize H as a function of e to get

Substituting from (8) and (12) in (1), we obtain

$$P = \frac{A_2}{\lambda}(1 - \bar{e}^{\lambda T}) + \frac{B_2}{\beta}(e^{\beta T} - 1) + \frac{C_2}{(\lambda+\beta)}\{1 - \bar{e}^{(\lambda+\beta)T}\}$$

$$- \frac{D_2}{(\lambda+2\beta)}\{e^{(\lambda+2\beta)T} - 1\}, \qquad (16)$$

where

$$A_2 = p\alpha - p\beta L_1 + K_1(p\gamma - k) - bK_1^2 \qquad (17)$$

$$B_2 = pB_1 + \frac{(\lambda+2\beta)}{\gamma}\{2b\,K_1B_1 - (p\gamma - k)\,B_1\} \qquad (18)$$

$$C_2 = p\,C_1 \qquad (19)$$

and $$D_2 = \frac{b(\lambda+2\beta)^2\,B_1^2}{\gamma^2} \qquad (20)$$

3.4.2 USE OF CALCULUS OF VARIATIONS

Substituting from (2) for e in (1) we have to maximize

$$P = \int_0^T \bar{e}^{\lambda t}\left[p(-\dot{x}) - \frac{b}{\gamma^2}(\alpha+\beta x+\dot{x})^2 + \frac{k}{\gamma}(\alpha+\beta x+\dot{x}) - \ell\right]dt \qquad (21)$$

Using Euler-Lagrange's equation of calculus of variations and simplifying we get

$$\ddot{x} - \lambda\dot{x} - \beta(\lambda+\beta)x + \frac{(\lambda+\beta)(p\gamma - 2b\alpha) - \lambda p\gamma^2}{2b} = 0 \qquad (22)$$

Integrating (22) with the boundary conditions $x(0) = R$ and $x(T) = 0$, we get the same equation for $x(t)$ as in (12). Substituting this value for $x(t)$ in (2), we get the same expression for $e(t)$ as in (8) and putting these two values in (1) and simplifying we get the same expression for P as in (16).

3.4.3 DISCUSSION OF SOLUTION

The expressions (8) , (12) and (16) are true for any arbitrary set of positive values of parameters. We study their variation with respect to t and T for the following set of parameters

$\alpha = 1.0$, $\beta = 1.0$, $\lambda = .1$, $\gamma = 4.0$, $k = 1.0$, $p = 500.0$, $b = 1$ and $x(0) = 100.0$

Fig. 3.1 shows the variation of $x(t)$ with respect to t with

$T = .035, .090, .290, .350$

Figs. 3.2, 3.3, 3.4 and 3.5 show the variation of $e(t)$ with respect to t for

$T = .035, .090, .290$ and $.350$

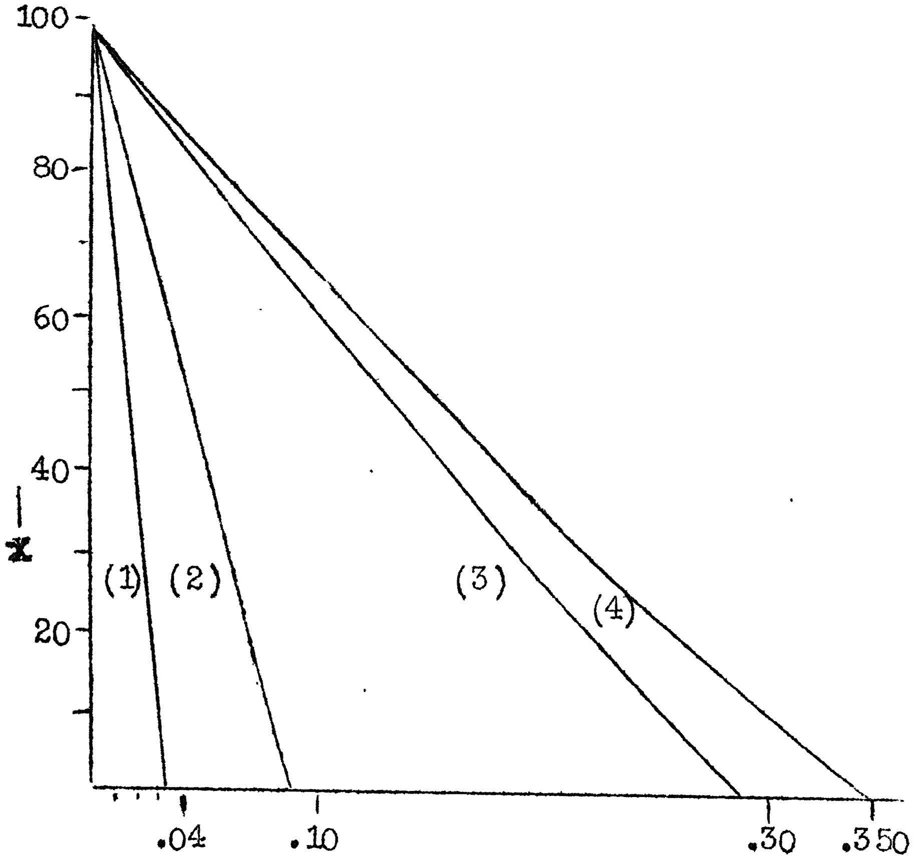
respectively.

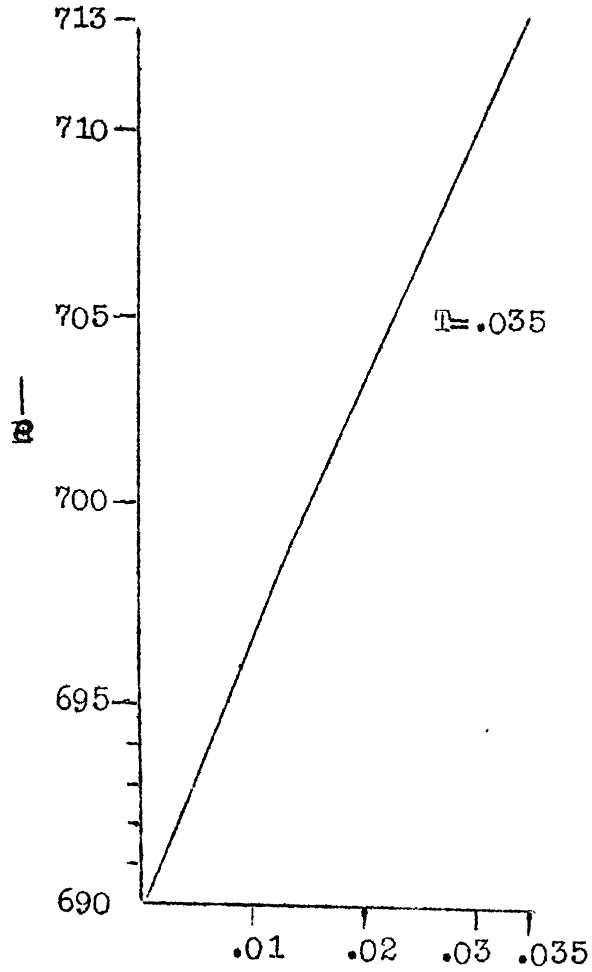
Fig. 3.6 shows the variation of P with respect to T.

Fig. 3.1 shows that with the increase in T, the rate of decrease of $x(t)$ slows down which is obvious. If we take T very small, $x(t)$ decreases very rapidly almost linearly.

Figs. 3.2 and 3.3 show that for exploiting the mine in a short interval we go on increasing the effort while on the other hand exploitation in a large interval leads to continuous decrease in the effort Figs. 3.4 and 3.5.

Fig. 3.6 shows that if we exploit the mine in a time interval which is smaller than a certain limit we go through a loss. But exploitation beyond this limit leads to the


100
80
60
40
20
X
(1)
(2)
(3)
(4)
.04
.10
.30
.350
t →
Fig. 3.1


713
710
705
700
695
690
T=.035
.01
.02
.03
.035
t →
Fig. 3.2

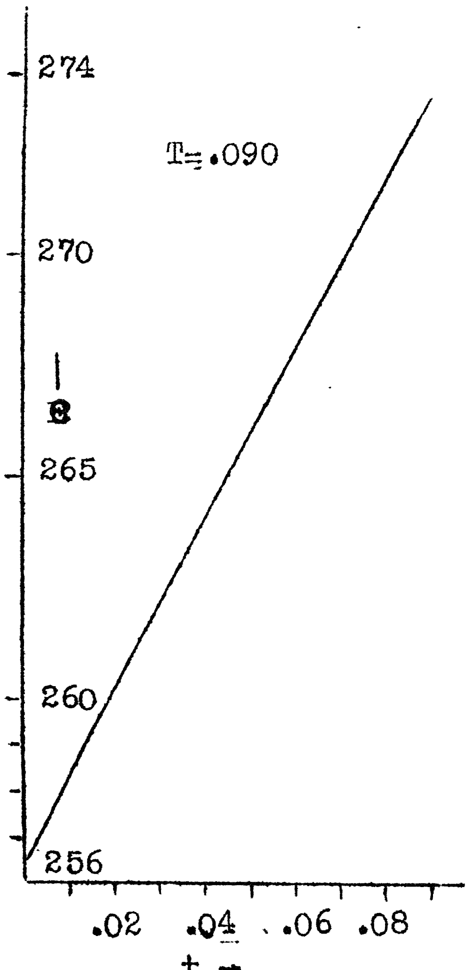


Fig. 3.3

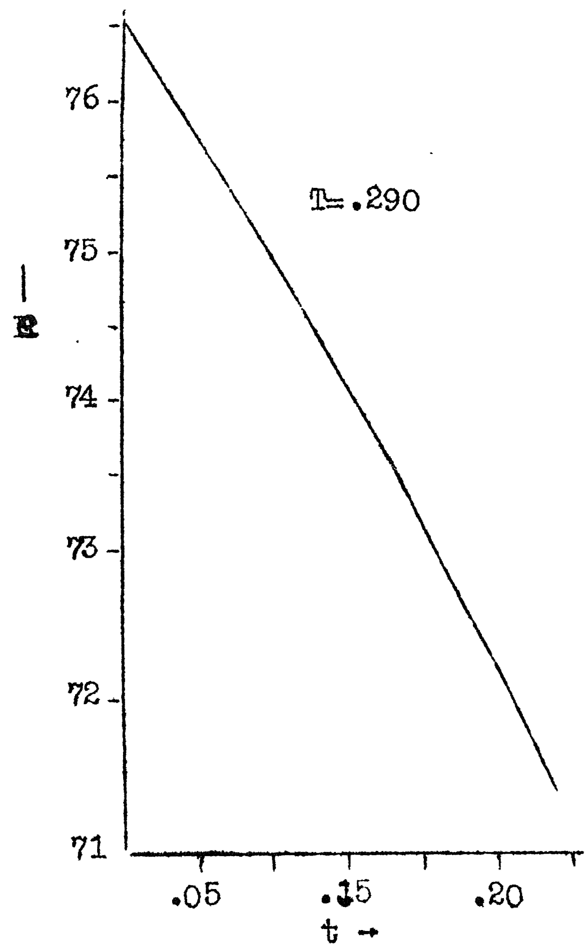


Fig. 3.4

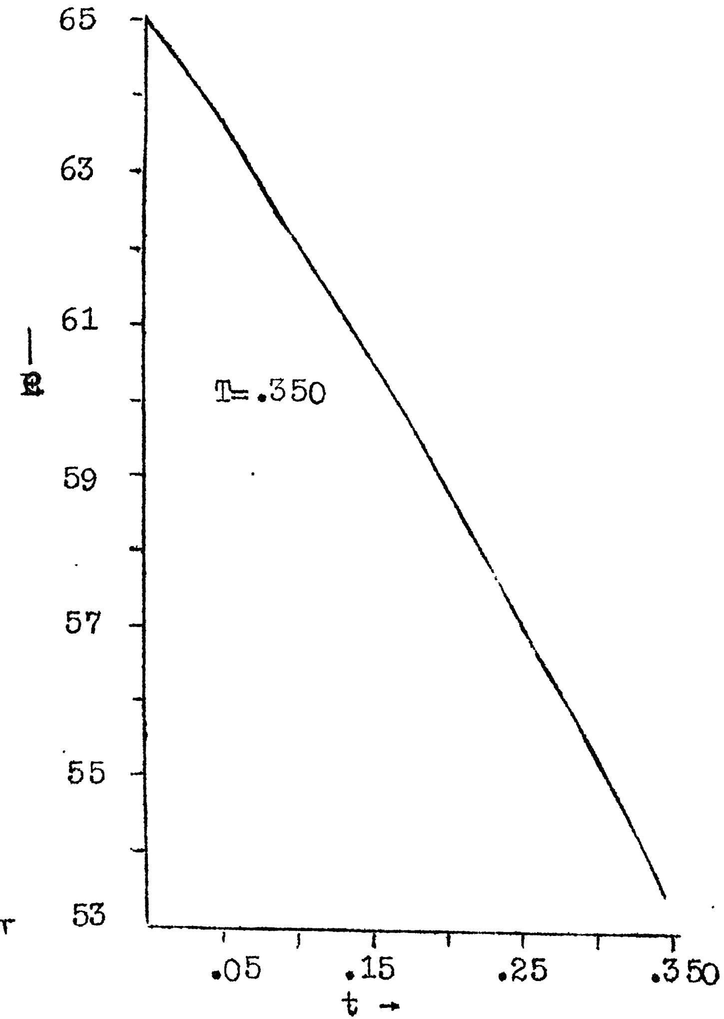


Fig. 3.5

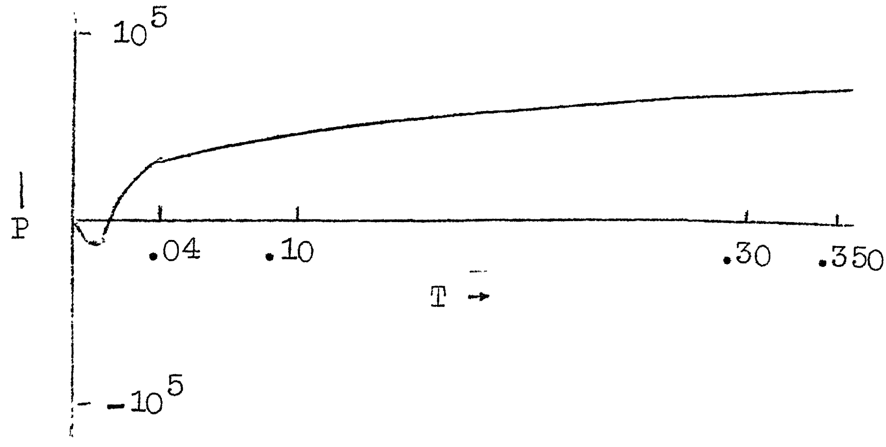


Fig. 3.6

profit and this profit goes on increasing with T. The figure also illustrates that we do not get any additional profit if we exploit the mine in a very large interval.

3.5 OPTIMIZATION FOR THE SECOND FUNCTION

For second function (Section 3.3, (2)), we have to maximize

$$P = \int_0^T \bar{e}^{\lambda t} \left[p(\alpha+\beta x+\gamma x^2+\delta e) - be^2 - ke - \ell \right] dt \tag{1}$$

subject to

$$\frac{dx}{dt} = - \left[\alpha + \beta x(t) + \gamma x^2(t) + \delta\, e(t) \right] \tag{2}$$

3.5.1 USE OF CALCULUS OF VARIATIONS

Substituting for e(t) from (2) in (1), we get

$$P = \int_0^T \bar{e}^{\lambda t} \Big[-p\dot{x} - \frac{b}{\delta^2} (\alpha + \beta x + \gamma x^2 + \dot{x})^2$$
$$+ \frac{k}{\delta} (\alpha+\beta x + \gamma x^2 + \dot{x}) - \ell \Big] dt \tag{3}$$

which is to be maximized. Using Euler-Lagrange's equation of calculus of variations and simplifying, we get

$$\ddot{x} - \lambda\dot{x} - 2\gamma^2 x^3 - \gamma(\lambda+3\beta)x^2 + \left[\frac{\gamma}{b} (k\delta - 2b\alpha) - \beta(\lambda+\beta) \right] x$$
$$+ \frac{(\lambda+\beta)(k\delta - 2b\alpha) - \lambda p \delta^2}{2b} = 0 \tag{4}$$

which has to be integrat~~ing~~ed subject to initial and final conditions

$$x(0) = R \text{ and } x(T) = 0, \tag{5}$$

where R is the total amount of material in the mine.

3.5.2 PROCEDURE FOR SOLVING THE PROBLEM

We have got to solve a second order non-linear equation in x which cannot be analytically integrated. We have here to solve a two-point boundary value problem. We integrate (4) subject to the initial conditions

$$x(0) = R, \ \dot{x}(0) = v_o \qquad (6)$$

and continue the numerical integration till the material in the mine is completely exhausted i.e.,till such time T where

$$x(T) = 0 \qquad (7)$$

For each value of v_o, we get in general a unique value of T. We try various values of v_o till we get the value of T given to us.

In practice we would like to treat T also as a control variable. We calculate $x(t)$ and $\frac{dx}{dt}$ for various values of v_o and for each v_o, we find T.

Knowing $x(t)$, $\frac{dx}{dt}$ and T, we can find $e(t)$ from (2) and then find P from (1). For each v_o there is a unique value of T and unique value of P. We can choose v_o so as to maximize P. Then we can choose the optimal period of exploiting the mine so that if choose this period and use the optimal mining policy, we shall get the maximum of all the profits.

3.5.3 DIMENSIONLESS FORM

Making the substitution

$$\frac{dx}{dt} = v \qquad (8)$$

equation (4) reduces to two first-order simultaneous differential equations in x and v as follows

$$\frac{dx}{dt} = v$$

$$\frac{dv}{dt} = \lambda v + 2\gamma^2 x^3 + \gamma(\lambda+3\beta)x^2 - \left[\frac{\gamma}{b}(k\delta - 2b\alpha) - \beta(\lambda+\beta)\right]x - \frac{(\lambda+\beta)(k\delta - 2b\alpha) - \lambda p\delta^2}{2b} \qquad (9)$$

Using the non-dimensional variables

$$X = \frac{\beta x}{\alpha}, \quad V = \frac{v}{\alpha} \quad \text{and} \quad \tau = t\beta, \qquad (10)$$

the equation (9) reduces to non-dimensional form

$$\frac{dX}{d\tau} = V$$

$$\frac{dV}{d\tau} = AV + BX^3 + (C_1 + 3C_2)X^2 - DX + F, \qquad (11)$$

where

$$A = \frac{\lambda}{\beta}, \quad B = \frac{2\gamma^2\alpha^2}{4}, \quad C_1 = \frac{\gamma\lambda\alpha}{3}$$

$$C_2 = \frac{\gamma\alpha}{\beta^2}, \quad D = \frac{\gamma(k\delta - 2b\alpha) - b\beta(\lambda+\beta)}{b\beta^2} \qquad (12)$$

$$F = \frac{\lambda p\delta^2 - (\lambda+\beta)(k\delta - 2b\alpha)}{2b\alpha\beta}$$

Equation (2) becomes

$$V(\tau) = \frac{dX}{d\tau} = -1 - X - C_2 X^2 - E(\tau), \qquad (13)$$

where $E(\tau) = \frac{\delta}{\alpha} e(t)$.

3.5.4 A NUMERICAL ILLUSTRATION

We integrate (11) for the following values of parameters

$$A = 0.1,\ B = 2.0,\ C_1 = 0.1,\ C_2 = 1.0$$
$$D = 2.1,\ F = 50.55, \tag{14}$$

and for the initial value

$$X(0) = 100.0 \tag{15}$$

From (13)

$$V(0) = -1 - X(0) - X^2(0) - E(0)$$
$$= -10101.0 - E(0) \tag{16}$$

We integrate

$$\frac{dX}{d\tau} = V$$

$$\frac{dV}{d\tau} = 0.1V + 2.0X^3 + 3.1X^2 - 2.1X + 50.55$$

subject to

$$X(0) = 100,\ V(0) = -10102,\ -10201,\ -10501, -11101, \quad \ldots (17)$$

for which $E(0) = 1,\ 100,\ 400,\ 1000.$

For each of the four cases, we find the optimal profit to get the following table

	E(0)	V(0)	βT	P
(a)	1	-10102	.147	98970.60
(b)	100	-10201	.040	67493.40
(c)	400	-10501	.025	5980.64
(d)	1000	-11101	.017	-98044.00

In case (a) initial rate of exploitation of mine is least, the time taken to exploit the mine fully is largest and

the present value of discounted profits is maximum. As the initial rate of exploitation increases, the total time of exploitation and the profits decrease fast and a stage is reached when the profit becomes zero and if we exploit the mine still faster, the exploitation of mine would lead to a loss, though due to use of calculus of variations, the losses should be minimum.

Fig. 3.7 shows the variation of $X(\tau)$ with τ for the four cases.

Fig. 3.8 shows the variation of $\frac{dX}{d\tau}$ with τ for the four cases.

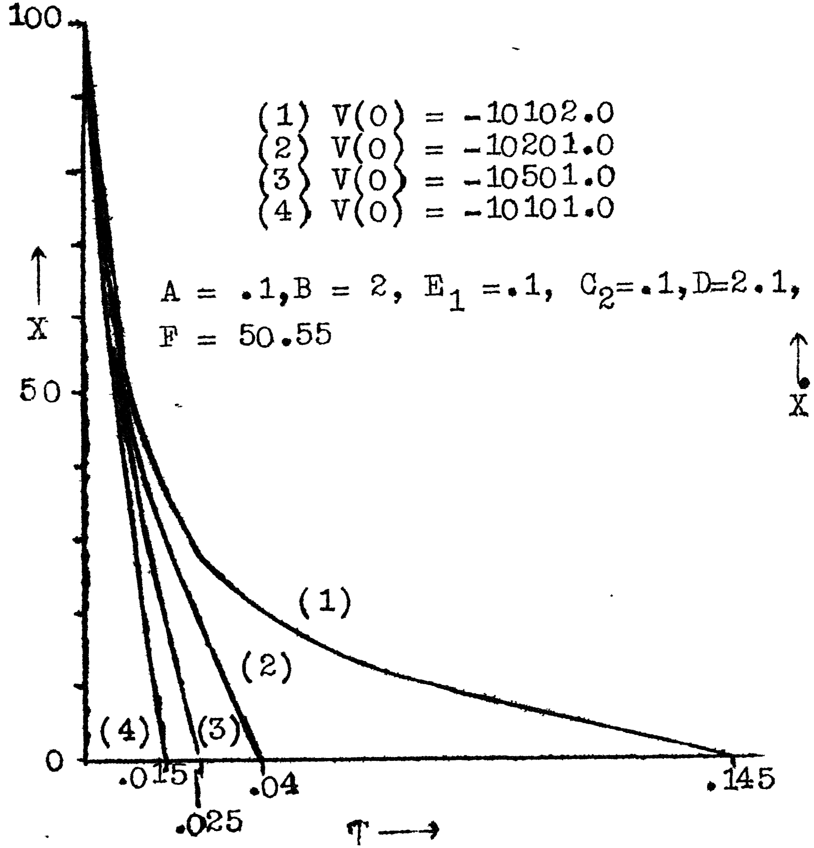
Figures 3.9, 3.10 and 3.11 show the variation of $E(\tau)$ with τ in the first three cases.

In the first case, the effort starts with initial value 1 and increases steadily to the value 130 over the time interval .147.

In the second case, the effort starts with the larger initial value of 100 and increases rather guickly to 1420 over the time interval .040.

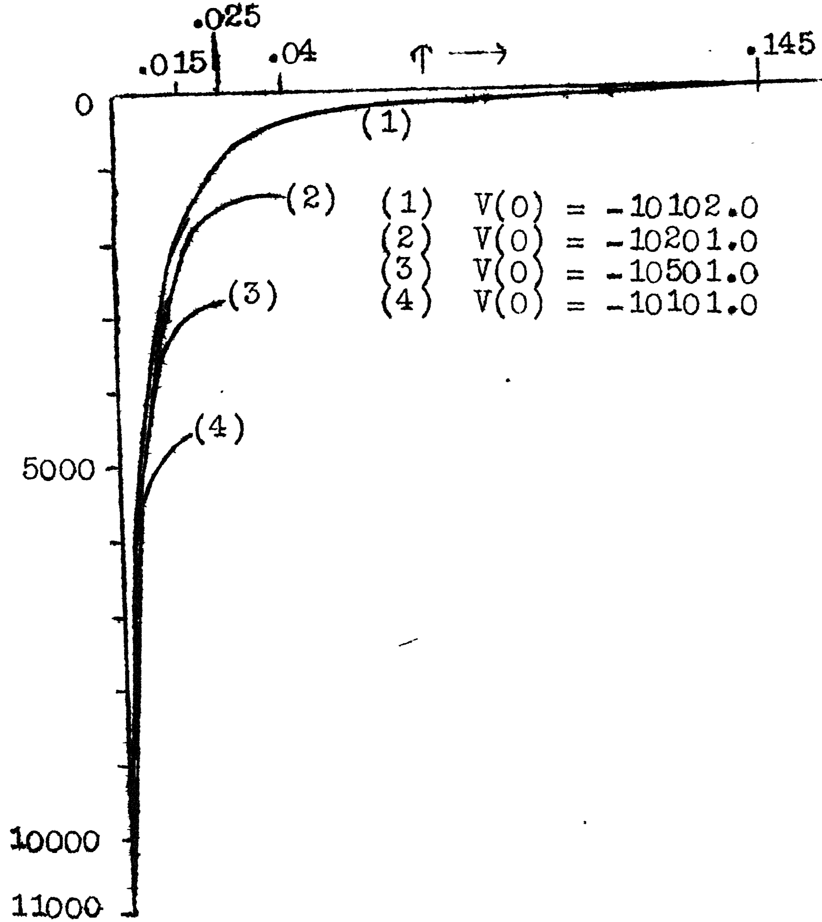
In the third case, the effort starts with the largest initial value of 400 and increases still more quickly to 2940 over the time interval .025.

In this example it appears that slower exploitation of mine over a long period is more profitable than quicker exploitation over short time.



100
50
0
X
(1) V(0) = -10102.0
(2) V(0) = -10201.0
(3) V(0) = -10501.0
(4) V(0) = -10101.0
A = .1, B = 2, E1 = .1, C2 = .1, D=2.1,
F = 50.55
(1)
(2)
(3)
(4)
.015
.025
.04
.145
T

Fig. 3.7



.015
.025
.04
.145
T
0
5000
10000
11000
X
(1)
(2)
(3)
(4)
(1) V(0) = -10102.0
(2) V(0) = -10201.0
(3) V(0) = -10501.0
(4) V(0) = -10101.0

Fig. 3.8

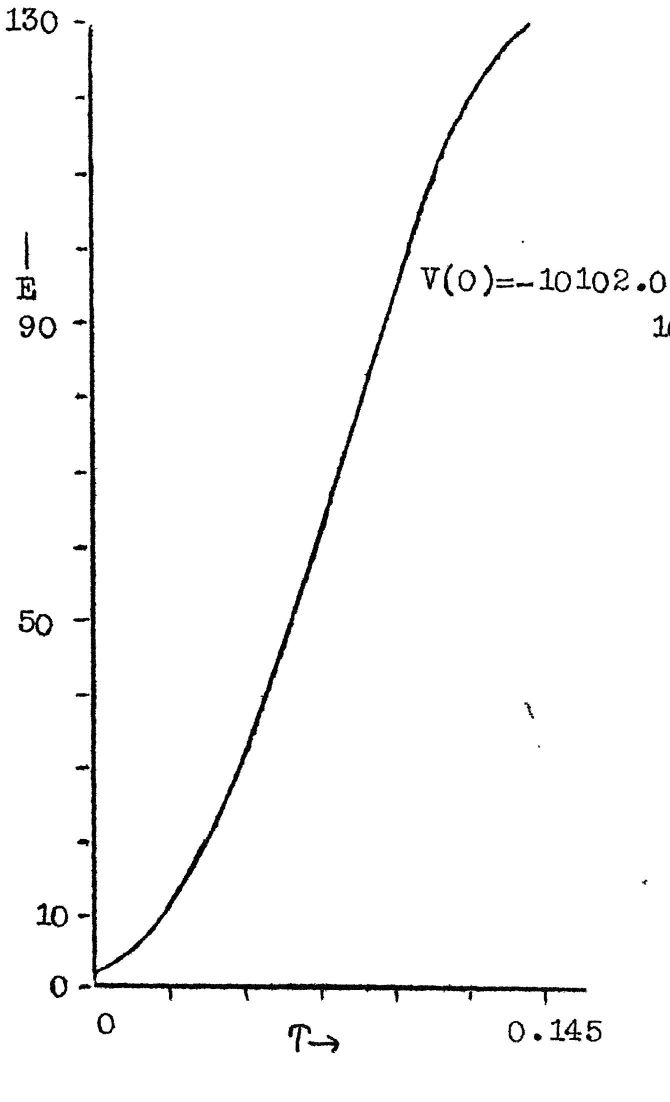


Fig. 3.9

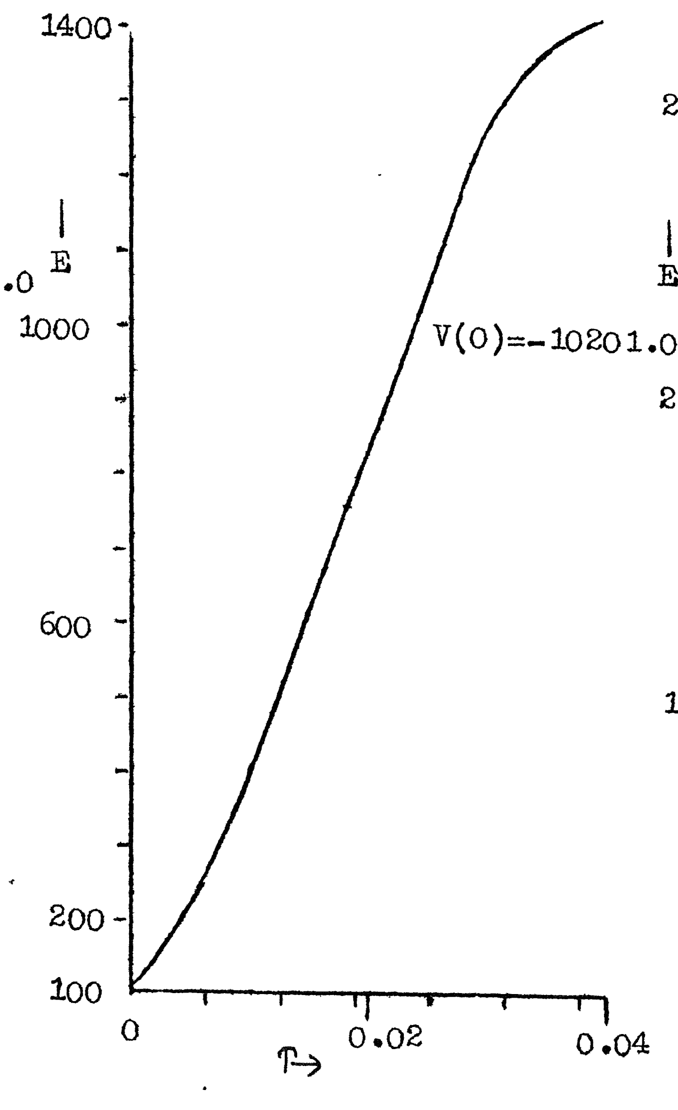


Fig. 3.10

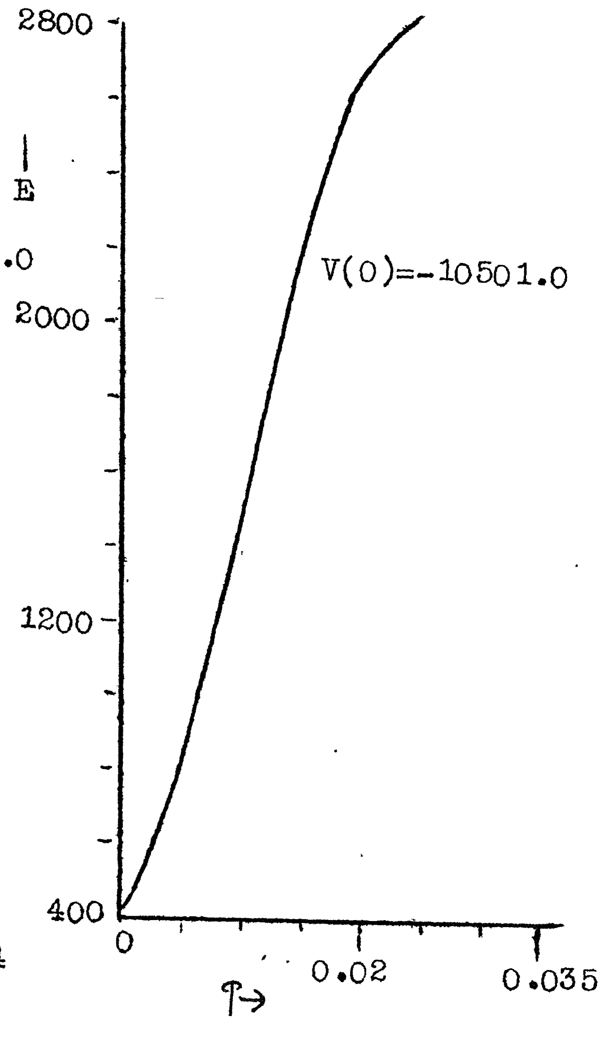


Fig. 3.11

3.6 OPTIMIZATION FOR THE THIRD FUNCTION

For the third function (Section 3.3., (3)), our problem is to maximize

$$P = \int_0^T \bar{e}^{\lambda t} \left[p\beta\, x(t)e(t) - be^2(t) - ke(t) - \ell \right] dt \qquad (1)$$

subject to

$$\frac{dx}{dt} = -\beta\, x(t)\, e(t) \qquad (2)$$

3.6.1 USE OF CALCULUS OF VARIATIONS

Substituting for e(t) from (2) in (1), we get

$$P = \int_0^T \bar{e}^{\lambda t} \left[p(-\dot{x}) - \frac{b\dot{x}^2}{\beta^2 x^2} + \frac{k\dot{x}}{\beta x} - \ell \right] dt, \qquad (3)$$

which is to be maximized. Using Eulr-Lagrange's equation of calculus of variations and simplifying, we get

$$\ddot{x}x - \dot{x}^2 - \lambda\dot{x}x + \frac{k\lambda\beta}{2b} x^2 - \frac{\lambda p\beta^2}{2b} x^3 = 0 \qquad (4)$$

which is to be integrated subject to the initial and final conditions

$$x(0) = R,\ x(T) = 0, \qquad (5)$$

where R is the total material in the mine.

3.6.2 HOW TO SOLVE THE PROBLEM

We have here to solve a second order non-linear differential equation in x which can not be integrated analytica-lly. We have got here to solve a two point boundary value problem. We integrate (4) subject to the initial conditions

$$x(0) = R, \left.\frac{dx}{dt}\right|_0 = v_o \tag{6}$$

and continue the integration till the material in the mine is supposed to be exhausted i.e. till such time T where

$$x(T) = 0. \tag{7}$$

For each value v_o, we get in general a unique value of T. As in earlier cases, here also we would like to treat T as a control variable. We calculate x(t) and $\frac{dx}{dt}$ for various values of v_o and for each v_o, we find T. We can choose the optimal period of exploiting the mine so that if choose this period and use the optimal mining policy, we shall get the maximum of all the profits.

3.6.3 DIMENSIONLESS FORM

Making the substitution $\frac{dx}{dt} = v$, equation (4) reduces to two first order simultaneous differential equations in x and v as follows

$$\frac{dx}{dt} = v$$

$$\frac{dv}{dt} = \frac{v^2}{x} + \lambda v - \frac{\beta\lambda k}{2b} x + \frac{\lambda p \beta^2}{2b} x^2 \tag{8}$$

Using the non-dimensional variables

$$V = \frac{v}{\beta}, \quad \tau = t\beta \tag{9}$$

equations (55) reduce to non-dimensional form

$$\frac{dx}{d\tau} = V$$

$$\frac{dV}{d\tau} = \frac{V^2}{x} + AV - Bx + Cx^2, \tag{10}$$

where $\quad A = \frac{\lambda}{\beta}$, $B = \frac{\lambda k}{2b\beta}$ (11)

and $\quad C = \frac{\lambda p}{2b}$.

Equation (49) gives

$$V(\tau) = -e(\tau)\, x(\tau) \tag{12}$$

3.6.4 NUMERICAL ILLUSTRATION

We integrate (10) for the following values of the parameters

$$A = 0.1,\ B = .25,\ C = 5.0 \tag{13}$$

and for initial value

$$x(0) = 10.0 \tag{14}$$

From (59) $\quad V(0) = -e(0)\, x(0)$

$$= -10\, e(0) \tag{15}$$

We integrate

$$\frac{dx}{d\tau} = V,\ \frac{dV}{d\tau} = \frac{V^2}{x} + .1\, V - .25\, x + 5\, x^2 \tag{16}$$

subject to

$$x(0) = 10.0,\ V(0) = -100,\ -150,\ -200 \tag{17}$$

for which e(0) = 10, 15, 20.

For each of the three cases, we find the optimal profit to get the following table :

	e(0)	V(0)	βT	P
(a)	10	-100	1.177	933.001
(b)	15	-150	0.375	904.343
(c)	20	-200	0.254	879.393

Our condition on $e(\tau)$ is to be positive for all times till the mine is exhausted. For all values of V_0 which are ~~less~~ greater than -100.0, $e(\tau)$ comes out to be negative some time before the mine is exhausted. That is why we start with $V(0) = -100.0$.

In this example, in case (a) initial rate of exploitation of mine is least, the time taken to exploit the mine till the material left in the mine is 1 percent is largest and the present value of discounted profits is maximum. As the initial rate of exploitation increases, the total time of exploitation and profit decrease.

Fig. 3.12 shows the variation of $x(\tau)$ with τ for the three cases.

Fig. 3.13 shows the variation of $\frac{dx}{d\tau}$ with τ for the three cases.

Fig. 3.14 shows the variation of $e(\tau)$ with τ in the three cases.

In the first case, the effort starts with initial value 10 and decreases steadily to the value 2.74506 in the time interval 1.156 then increases to the value 2.74540 till the material left in mine is 1.

In the second case, the effort starts with the larger initial value 15 and decreases rather slowly to 11.7706 in the time-interval .288 then increases to the value 11.8199.

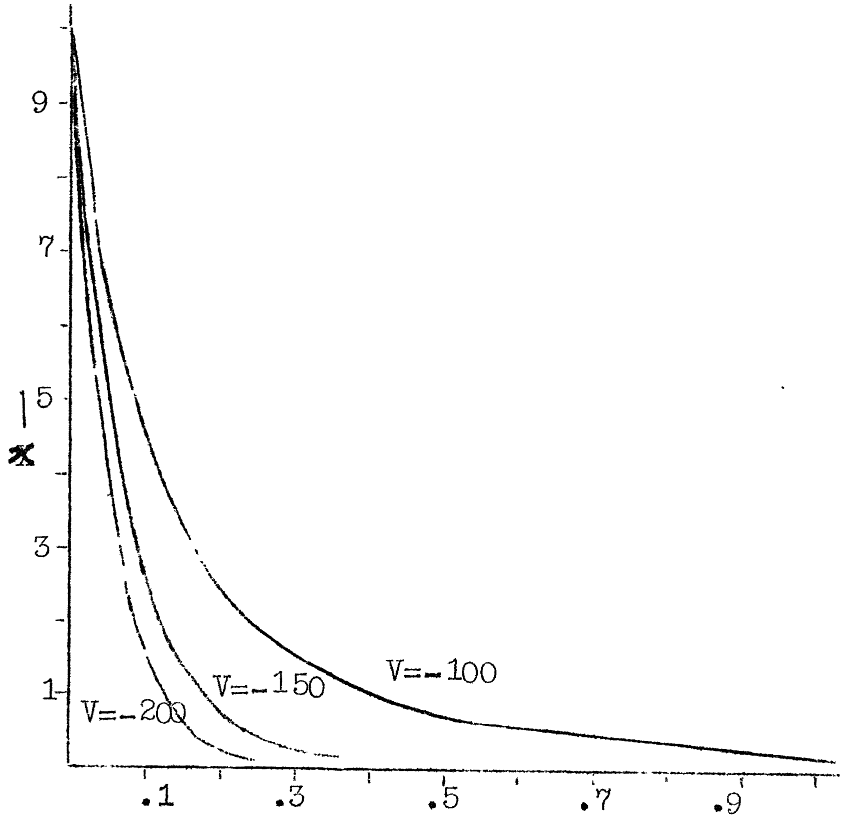


Fig. 3.12

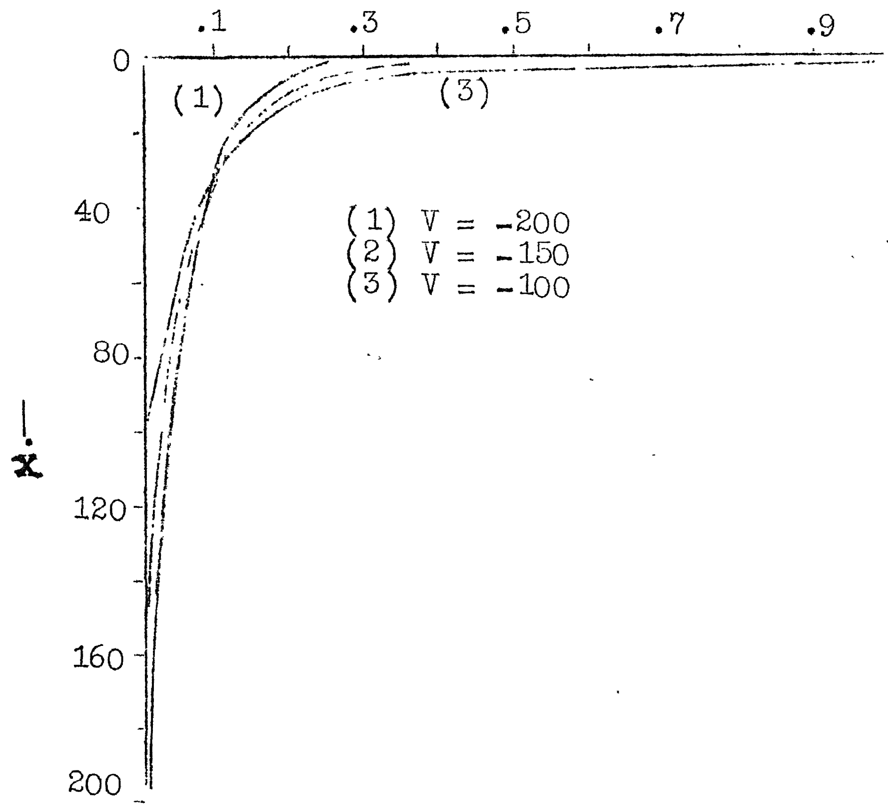


Fig. 3.13

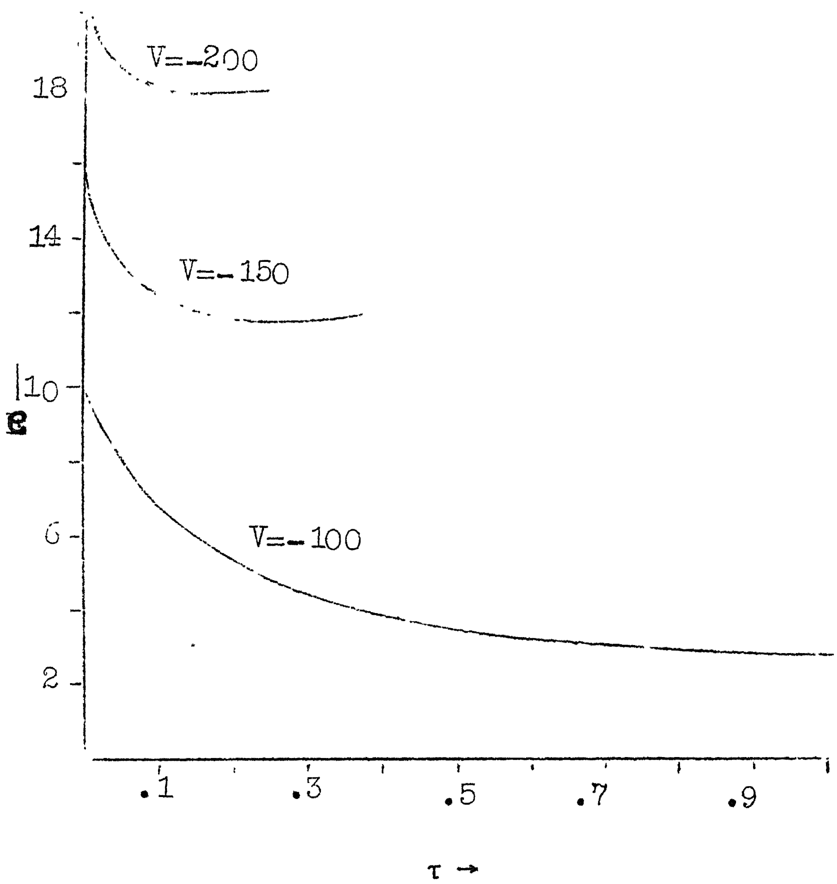


Fig. 3.14

In the third case, the effort starts with the largest value 20 and decreases still more slowly to the value 17.8161 in the time interval .175 and then increases to the value 17.891.

In this example also it appears that slower exploitation of mine over a long period is more profitable than the quicker exploitation over short-time.

CHAPTER IV

BIRTH AND DEATH PROCESSES WITH MIGRATION

This chapter runs over two sections

4.1 Continuous-time linear birth-death-emigration process.

4.2 Discrete-time linear birth-death-immigration-emigration process.

In continuous time birth-death-emigration process, we obtain the probability of extinction numerically, in terms of the solution of an Abel's integral equation, which has been left out by Getz [1975] and pointed out by Kapur [1979a].

In discrete-time linear birth-death-immigration emigration process, a discrete version of the continuous time birth-death-immigration-emigration process is given and solved numerically.

4.1 CONTINUOUS-TIME LINEAR BIRTH-DEATH EMIGRATION PROCESS

4.1.1 INTRODUCTION. Let n denote the size of population and $p(n,t)$ the probability of being n persons in the system at any time t. Let the probability of a birth, a death, an immigration and an emigration be $n\lambda\Delta t + O(\Delta t)$, $n\mu\Delta t + O(\Delta t)$, $\nu\Delta t + O(\Delta t)$ and $\alpha\Delta t + O(\Delta t)$. respectively. We obtain the following system of differential-difference equations

$$P'(n,t) = [(n+1)\mu+\alpha]P(n+1,t) - [n(\mu+\lambda)+\alpha+\nu]\ P(n,t)$$

$$+ [(n-1)\lambda+\nu]\ P(n-1,t) \quad n=1,2,3,\ldots \tag{1}$$

$$P'(0,t) = (\mu+\alpha)P(1,t) - \nu P(0,t) \tag{2}$$

Defining the probability-generating function

$$G(s,t) = \sum_{n=0}^{\infty} P(n,t)\ s^n, \tag{3}$$

multiplying (1) by s^n and (2) by s^o and summing up for all values of n, we get the partial differential equation

$$\frac{\partial G}{\partial t} = (\lambda s-\mu)(s-1)\frac{\partial G}{\partial s} + (\nu - \frac{\alpha}{s})(s-1)\ G + \frac{\alpha(s-1)}{s} P(0,t) \tag{4}$$

Getz [1975,76] obtained (4) without the last term on the right hand side and solved it, but the omission of this term containing the probability of extinction P(0,t) made all his calculations wrong. This error was pointed out in Kapur [1979a] and was accepted by Getz [1978]. The problem of solving the complete equation (4) however remained. For the steady case this was accomplished in Kapur [1978a,78b,78c,78d,79d]. For the non-steady case, we obtain the solution below for the case $\nu = 0$, $\alpha < \mu$ in terms of the solution of an Abel's integral equation. Another solution in terms of incomplete beta functions has been given elsewhere Kapur [1979b, 1979c]. As expected the agreement between the two solution is quite good.

4.1.2 BASIC INTEGRAL EQUATION

If n_o denotes the number of persons initially in the system, the solution of (4) is found out in Kapur [1979a].

$$\text{Let } P(0,t) = \beta(t) \tag{5}$$

Then equation (4) becomes

$$\frac{\partial G}{\partial t} = (\lambda s-\mu)(s-1)\frac{\partial G}{\partial s} + \left(\nu - \frac{\alpha}{s}\right)(s-1)\, G + \frac{\alpha(s-1)}{s}\beta(t) \tag{6}$$

The auxiliary equations are

$$\frac{dt}{1} = \frac{ds}{(\lambda s-\mu)(1-s)} = \frac{dG}{\left(\nu - \frac{\alpha}{s}\right)(s-1)\, G + \frac{\alpha(s-1)}{s}\beta(t)} \tag{7}$$

One intermediate integral is

$$\frac{\lambda s - \mu}{1-s}\, e^{-(\lambda-\mu)t} = \frac{\lambda s - \mu}{1-s}\, z = K \tag{8}$$

For the second intermediate integral

$$\frac{ds}{(\lambda s-\mu)(1-s)} = \frac{dG}{\left(\nu - \frac{\alpha}{s}\right)(s-1)G + \frac{\alpha(s-1)}{s}\beta(t)}, \tag{9}$$

we get the equation

$$\frac{dG}{dz} - \frac{(\mu\nu - \alpha\lambda)\, z + K(\nu-\alpha)}{(\mu z + K)\,(\lambda z + K)}\, G = \frac{\alpha}{\mu z + K}\, f(z)$$

$$= \frac{\alpha}{\mu z + K}\, \beta\left(\frac{\log z}{\mu - \lambda}\right) \tag{10}$$

Integrating and using (8),

$$G(s,t)\left[\frac{(\lambda-\mu)s}{(1-s)}z\right]^{\frac{\alpha}{\mu}}\left[\frac{\lambda-\mu}{1-s}z\right]^{-\nu/\lambda}$$

$$=\alpha\left[\int_1^z f(z)(\mu z+K)^{\frac{\alpha}{\mu}-1}(\lambda z+K)^{-\nu/\lambda}\,dz\right]_{K=\frac{\lambda s-\mu}{1-s}z}$$

$$+\ \psi\left(\frac{\lambda s-\mu}{1-s}z\right) \qquad (11)$$

where $\psi(u)$ is an arbitrary function of u. If initially there are n_0 persons in the system, we get

$$s^{n_0}\left[\frac{\lambda-\mu}{1-s}\right]^{\frac{\alpha}{\mu}-\frac{\nu}{\lambda}} s^{\frac{\alpha}{\mu}} = \psi\left[\frac{\lambda s-\mu}{1-s}\right] \qquad (12)$$

giving

$$\psi(y) = \frac{(y+\mu)^{n_0+\frac{\alpha}{\mu}}}{(y+\lambda)^{n_0+\nu/\lambda}} \qquad (13)$$

So that

$$G(s,t)\left(\frac{\lambda-\mu}{1-s}\right)^{\frac{\alpha}{\mu}-\frac{\nu}{\lambda}} z^{\frac{\alpha}{\mu}-\frac{\nu}{\lambda}} s^{\frac{\alpha}{\mu}} =$$

$$\alpha\left[\int_1^z f(z)(\mu z+K)^{\frac{\alpha}{\mu}-1}(\lambda z+K)^{-\frac{\nu}{\lambda}}\,dz\right]_{K=\frac{\lambda s-\mu}{1-s}z}$$

$$+\ \frac{\left(\frac{\lambda s-\mu}{1-s}z+\mu\right)^{n_0+\frac{\alpha}{\mu}}}{\left(\frac{\lambda s-\mu}{1-s}z+\lambda\right)^{n_0+\nu/\lambda}} \qquad (14)$$

Putting s = 0, we get in (14) and simplifying, we get

$$\int_1^x f(z)\ (\mu z - \mu x)^{\frac{\alpha}{\mu} - 1}\ (\lambda z - \mu x)^{-\nu/\lambda}\ dz$$

$$= -\frac{1}{\alpha}\ \frac{(\mu - \mu x)^{n_o + \alpha/\mu}}{(\lambda - \mu x)^{n_o + \nu/\lambda}} \qquad (15)$$

This is Volterra's integral equation of the first kind whose solution should determine $P(0,t)$.

Putting $\nu = 0$, we get Abel's integral equation

$$\int_1^x f(z)\ (z-x)^{k-1}\ dz = -\frac{1}{k}\ \frac{(1-x)^{n_o+k}}{(K-x)^{n_o}}, \qquad (16)$$

where $k = \frac{\alpha}{\mu}$ and $K = \lambda/\mu$ (17)

4.1.3 <u>SOLUTION OF INTEGRAL EQUATION WHEN $\lambda < \mu$</u>

It is known Myskis [1975] that when $0 < a < 1$ the solution of

$$\int_b^x \frac{1}{(x-z)^a}\ f(z)\ dz = F(x), \quad [F(b) = 0] \qquad (18)$$

is given by

$$f(x) = \frac{\sin \pi a}{\pi} \int_b^x \frac{F'(y)dy}{(x-y)^{1-a}} \qquad (19)$$

When $\lambda < \mu,\ z > 1,\ x > 1,\ 1 < z < x$ (20)

Equation ~~(9)~~ (16) can be written as

$$\int_1^x \frac{f(z)dz}{(x-z)^{1-k}} = \frac{1}{k}\ \frac{(x-1)^{n_o+k}}{(x - K)^{n_o}} \qquad (21)$$

When $0 < k < 1$, its solution is

$$f(x) = \frac{\sin \pi(1-k)}{\pi} \frac{1}{k} \int_1^x \frac{\frac{d}{dy} \frac{(y-1)^{n_o+k}}{(y-K)^{n_o}}}{(x-y)^k} dy$$

$$= \frac{\sin \pi(1-k)}{k\pi} \left[(n_o+k) \int_1^x \frac{(y-1)^{n_o+k-1}}{(y-K)^{n_o} (x-y)^k} dy \right.$$

$$\left. - n_o \frac{(y-1)^{n_o+k}}{(y-K)^{n_o+1} (x-y)^k} dy \right] \tag{22}$$

The two integrals on the R.H.S. have to be evaluated numerically.

4.1.4 SOLUTION OF THE INTEGRAL EQUATION WHEN $\lambda > \mu$

In this case $x < 1$, $z < 1$, $x < z < 1$, $K > 1$ and the solution is given by

$$f(x) = \frac{\sin \pi(1-k)}{k\pi} \left[(n_o+k) \int_x^1 \frac{(1-y)^{n_o+k-1}}{(K-y)^{n_o} (y-x)^k} dy \right.$$

$$\left. - n_o \int_x^1 \frac{(1-y)^{n_o+k}}{(K-y)^{n_o+1} (y-x)^k} dy \right] \tag{23}$$

The two integrals on the R.H.S. have to be evaluated numerically.

4.1.5 SOLUTION OF THE INTEGRAL EQUATION WHEN $\lambda = \mu$

In this case we cannot use the substitution $z = e^{-(\lambda-\mu)t}$.

Equation (4) can be written as

$$\frac{\partial G}{\partial t} = \mu(s-1)^2 \frac{\partial G}{\partial s} + (\nu - \frac{\alpha}{s})(s-1)G + \frac{\alpha(s-1)}{s}\beta(t) \tag{24}$$

Its solution is

$$G(s,t)\frac{s^{\alpha/\mu}}{(s-1)^{\frac{\alpha-\nu}{\mu}}} = \int_0^t \alpha\beta(T)\left(\mu T + \frac{1}{s-1} - \mu t\right)^{-\nu/\mu} \left(\mu T + \frac{1}{s-1} - \mu t + 1\right)^{\frac{\alpha}{\mu}-1} dT$$

$$+ \left(\frac{1}{s-1} - \mu t\right)^{-\nu/\mu}\left(\frac{1}{s-1} - \mu t + 1\right)^{\frac{\alpha}{\mu}}\left(\frac{1}{\frac{1}{s-1} - \mu t} + 1\right)^{n_o} \tag{25}$$

Putting $s = 0$, we get

$$\alpha\int_0^t \beta(T)(\mu t+1-\mu T)^{-\nu/\mu}(\mu t - \mu T)^{\frac{\alpha}{\mu}-1} dT$$

$$= \mu(1+\mu t)^{-\nu/\mu}\, t^{\alpha/\mu}\left(\frac{\mu t}{1+\mu t}\right)^{n_o} \tag{26}$$

which is again a Volterra's integral equation of the first kind.

When $\nu = 0$, we get

$$\int_0^t \frac{\beta(T)\alpha T}{(t-T)^{1-k}} = \frac{1}{k}\, t^k \left(\frac{\mu t}{1+\mu t}\right)^{n_o} \tag{27}$$

Its solution is

$$\beta(t) = \frac{\sin \pi(1-k)}{\pi k}\int_0^t \frac{kT^{k-1}\left(\frac{\mu T}{1+\mu T}\right)^{n_o} + T^k n_o\left(\frac{\mu T}{1+\mu T}\right)^{n_o-1} \frac{\mu}{(1+\mu T)^2}\, dT}{(t-T)^k}$$

$$= \frac{\sin \pi(1-k)}{\pi k} \left[\int_0^t \frac{kT^{k-1}}{(t-T)^k} \left(\frac{\mu T}{1+\mu T}\right)^{n_o} dT \right.$$

$$\left. + n_o \mu \int_0^t \frac{T^k (\mu T)^{n_o - 1}}{(t-T)^k (1+\mu T)^{n_o+1}} dT \right] \qquad (28)$$

The integrals on the R.H.S have to be evaluated numerically.

4.1.6 NUMERICAL INTEGRATION

We use the method of L.V. Kantorovich, Demidovich [1976], for isolating singularities to calculate the approximate value of the integrals as follows

$$\int_0^t \frac{\phi(T)\, dT}{(t-T)^{1-k}} = \int_0^t \frac{\phi(t-y)\, dy}{y^{1-k}}$$

$$= \frac{\phi(t)}{k} t^k - \frac{\phi'(t)}{\lfloor 1} \frac{t^{k+1}}{k+1} + \frac{\phi''(t)}{\lfloor 2} \frac{t^{k+2}}{k+2}$$

$$- \frac{\phi'''(t)}{\lfloor 3} \frac{t^{k+3}}{k+3} + \int_0^t \frac{\psi(t-y)\, dy}{y^{1-k}}$$

The integral on the R.H.S. is a proper integral and is computed by Simpson's formula. The results are illustrated in the following table. The numbers in brackets give the values obtained by using the expressions in terms of incomplete beta functions Kapur [1979b, 1979c] and the tables of incomplete beta functions Pearson [1968]. The small differences are due to approximate numerical integration in one case and linear interpolation in tables the others.

n_o	λ	μ	α	λ/μ	α/μ	t=1	t=2	t=3
2	2.5	2.0	1.0	1.25	.50	.545359 (.533592)	.724491 (.691823)	.794526 (.748388)
2	4.0	2.0	1.50	2.00	.75	.388384 (.384753)	.432895 (.426546)	.438593 (.440640)
2	1.0	1.0	.50	1.0	.50	.381369 (.381282)	.615164 (.605478)	.734427 (.734375)
2	1.0	1.0	.75	1.0	.75	.442578 (.440643)	.683179 (.668642)	.795613 (.789062)
2	1.0	2.0	1.0	.50	.50	.780588 (.776080)	.94872 (.946147)	.97889 (.987243)
2	3.0	4.0	3.0	.75	.75	.93611 (.92282)	.98988 (.987904)	.99582 (.99591)

4.2 DISCRETE-TIME BIRTH-DEATH-IMMIGRATION-EMIGRATION-PROCESS

4.2.1 THE BASIC EQUATIONS AND THEIR SOLUTION

The basic system of differential-difference equations for the continuous-time B-D-E-I process is given by

$$\left.\begin{aligned} P'(n,t) &= P(n-1,t)\left[(n-1)\lambda+\nu\right] - P(n,t)\left[n(\mu+\lambda)+\alpha+\nu\right] \\ &\quad + P(n+1,t)\left[(n+1)\mu+\alpha\right] \qquad n = 1,2,3,\ldots \\ P'(0,t) &= -\nu P(0,t) + (\mu+\alpha)\,P(1,t), \end{aligned}\right] \quad (1)$$

where $P(n,t)$ is the probability of there being n persons in the system at time t.

We now consider the discrete version of (1). viz.

$$\left.\begin{aligned} P(n,t+1) &= P(n-1,t)\left[(n-1)\lambda+\nu\right]+P(n,t)\left[1-n\mu-n\lambda-\alpha-\nu\right] \\ &\quad + P(n+1,t)\left[(n+1)\mu+\alpha\right] \quad n = 1,2,3,\ldots\left[\frac{1-\alpha-\nu}{\lambda+\mu}\right] \\ P(0,t+1) &= (1-\nu)\,P(0,t) + (\mu+\alpha)\,P(1,t), \end{aligned}\right] \quad (2)$$

We can regard (2) as the basic equations for a discrete B-D-E-I process. Given the probability distribution at time t, (2) will enable us to find the probability distribution at time $t+1$, so that if we know the probability distribution at time 0, we can find the probability distribution at future times.

If there are n_0 persons in the system at time 0,

$$P(n_0,0) = 1,\; P(n,0) = 0 \text{ when } n \neq n_0 \qquad (3)$$

Let $n_0 = 3$, then by using the above equations, we get

$$\left.\begin{array}{l} P(0,0) = 0,\ P(1,0) = 0,\ P(2,0) = 0,\ P(3,0) = 1,\ P(4,0) = 0 \\ P(5,0) = 0, \ldots \end{array}\right] \quad (4)$$

$$\left.\begin{array}{l} P(0,1) = 0, P(1,1)=0, P(2,1)=3\mu+\alpha, P(3,1)=1-3\mu-3\lambda-\alpha-\nu, \\ P(4,1) = 3\lambda+\nu,\ P(5,1) = 0,\ P(6,1) = 0, \ldots \end{array}\right] \quad (5)$$

$$\left.\begin{array}{l} P(0,2)=0, P(1,2)=(2\mu+\alpha)(3\mu+\alpha), P(2,2)=(3\mu+\alpha)(2-5\mu-5\lambda-2\alpha-2\nu), \\ P(3,2)=(1-3\mu-3\lambda-\alpha-\nu)^2+(3\mu+\alpha)(2\lambda+\nu)+(4\mu+\alpha)(3\lambda+\nu), \\ P(4,2)=(3\lambda+\nu)(1-4\mu-4\lambda-\alpha-\nu)+(1-3\mu-3\lambda-\alpha-\nu)(3\lambda+\nu), \\ P(5,2)=(3\lambda+\nu)(4\lambda+\nu);\ P(n,2) = 0 \text{ if } n > 5 \end{array}\right] \quad (6)$$

We can similarly find $P(n,t)$ for $t = 3,4,5,\ldots$ and n varying from 0 to $3+t$.

It is easily verified that

$$\left.\begin{array}{l} \sum_{n=0}^{\infty} P(n,1) = 1,\ \sum_{n=0}^{\infty} P(n,2) = 1 \\ (\bar{n})_{t=1} = \sum_{n=0}^{\infty} nP(n,1) = 3+3\lambda-3\mu-\alpha+\nu \\ (\bar{n})_{t=2} = \sum_{n=0}^{\infty} nP(n,2) = (3\mu+\alpha)(4-4\lambda-8\mu-\nu-3\alpha) \\ \qquad +(3\lambda+\nu)(8-8\lambda-16\mu-3\nu-5\alpha) \\ \qquad +3(1-3\lambda-3\mu-\alpha-\nu)^2 \end{array}\right] \quad (7)$$

4.2.2 SOME RESULTS ABOUT THE PROCESS

(i) If at $t = 0$, there are n_o persons in the system, then

$$P(n,t) = 0 \text{ if } n > n_o + t \text{ or } n < n_o - t \quad (8)$$

(ii) Thus the non-zero probabilities are :

$$\left.\begin{array}{l}
P(n_o,0) \\
P(n_o-1,1), P(n_o,1), P(n_o+1,1) \\
P(n_o-2,2), P(n_o-1,2), P(n_o,2), P(n_o+1,2), P(n_o+2,2) \\
------------------ \\
------------------ \\
P(0,n_o), P(1,n_o), P(2,n_o), \ldots\ldots\ldots, P(2n_o,n_o) \\
------------------ \\
------------------ \\
P(0,n), P(1,n), P(2,n), \ldots\ldots\ldots, P(n_o+n,n) \\

\end{array}\right] \quad (9)$$

where the sum of probabilities in each row is unit.

(iii) The number of non-zero probabilities upto time $n(\geq n_o)$ is

$$= [1+3+5+\ldots+(2n_o+1)] + [(2n_o+2)+(2n_o+3)+\ldots \ldots + (n_o+n+1)]$$

$$= \frac{1}{2}[n^2 + 2nn_o + 3n + n_o - n_o^2 + 2] \quad (10)$$

(iv) Using (8), equation (2) gives

$$P(t+1+n_o, t+1) = P(t+n_o, t)[(t+n_o)\lambda + \nu]$$

$$= P(t+n_o-1, t-1)[(t+n_o-1)\lambda+\nu][(t+n_o)\lambda+\nu]$$

$$= \ldots\ldots\ldots$$

$$= \left[[(t+n_0)\lambda+\nu][(t+n_0-1)\lambda+\nu] \cdots \right. \\ \left. \cdots [n_0\lambda+\nu] P(n_0,0), \right] \tag{11}$$

so that

$$P(t+n_0,t) = \prod_{m=n_0}^{n_0+t-1} (m\lambda+\nu) \tag{12}$$

We may note that this probability is independent of α and μ.

(v) Multiplying (2) by n and summing for all values of n, we get

$$(\bar{n})_{t+1} = (1+\lambda-\mu)\,\bar{n}(t) + \nu - \alpha + \alpha\, P(0,t), \tag{13}$$

which can be compared with

$$\frac{d\bar{n}}{dt} = (\lambda-\mu)\,\bar{n} + \nu - \alpha + \alpha\, P(0,t), \tag{14}$$

obtained in the continuous case [Kapur [1979a]].

(vi) Multiplying (2) by n^2 and summing for all values of n, we get

$$\left. \begin{aligned} (\bar{n}^2)_{t+1} &= (1+2\lambda-2\mu)(\bar{n}^2)_t + (2\mu-2\alpha+\lambda+\nu)\,\bar{n} \\ &\quad + \nu + \alpha - \alpha\, P(0,t), \end{aligned} \right] \tag{15}$$

which can be compared with

$$\left. \begin{aligned} \frac{d}{dt} E(n^2) &= 2(\lambda-\mu)E(n^2) + (2\nu-2\alpha+\lambda+\mu)\,\bar{n} \\ &\quad + \nu + \alpha - \alpha\, P(0,t), \end{aligned} \right] \tag{16}$$

obtained in the continuous case [Kapur [1979a]].

(vii) Getz [1975,1976] obtained (14) and (16) without the terms involving $P(0,t)$. These made the calculations of $(\bar{n})_t$ and $(\sigma^2)_t$ simpler, but the formulae obtained were incorrect. We cannot also use (13) and (15) to obtain $(\bar{n})_t$ and $(\sigma^2)_t$ unless we first obtain seperately the probability of extinction for all t.

4.2.3 GRAPHS

(i) In Figures 4.1 - 4.3, we have plotted $P(n,t)$ against n for $t = 1, 11, 21, 31$ for $\lambda = 0.3$, $\mu = 0.1$, $\alpha = .05$, $\nu = 0.02$ and $n_o = 3,4,5$ (17)

(ii) In Figures 4.4 - 4.6, we have plotted $P(n,t)$ against n for $t = 1, 11, 21, 31$ for $\lambda = 0.03$, $\mu = 0.1$, $\alpha = 0.0$, $\nu = 0.0$ and $n_o = 3,4,5$. (18)

(iii) In Figures 4.7 - 4.8, we have plotted $P(n,t)$ against t for $\lambda = 0.03$, $\mu = 0.1$, $\alpha = 0.05$, $\nu = 0.02$ and $n_o = 3,5$. (19)

(iv) In Figures 4.9 - 4.10, we have plotted $\bar{n}(t)$ and $\sigma^2(t)$ against t for

$\lambda = 0.03$, $\mu = 0.1$, $\alpha = 0.05$, $\nu = 0.02$ and $n_o = 3,5$. (20)

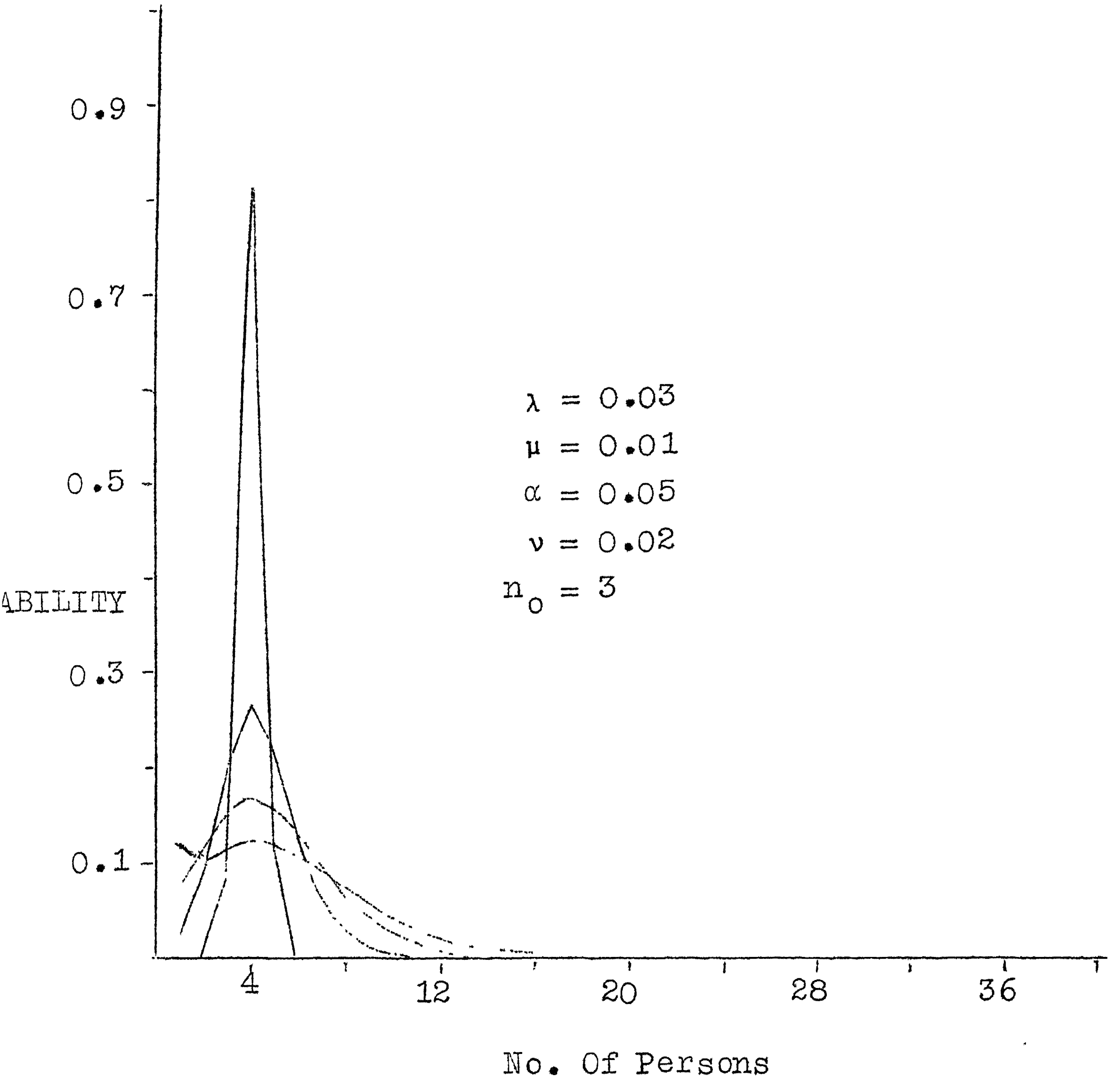

0.9
0.7
0.5
ABILITY
0.3
0.1
λ = 0.03
μ = 0.01
α = 0.05
ν = 0.02
n_o = 3
4
12
20
28
36
No. Of Persons

Fig. 4.1

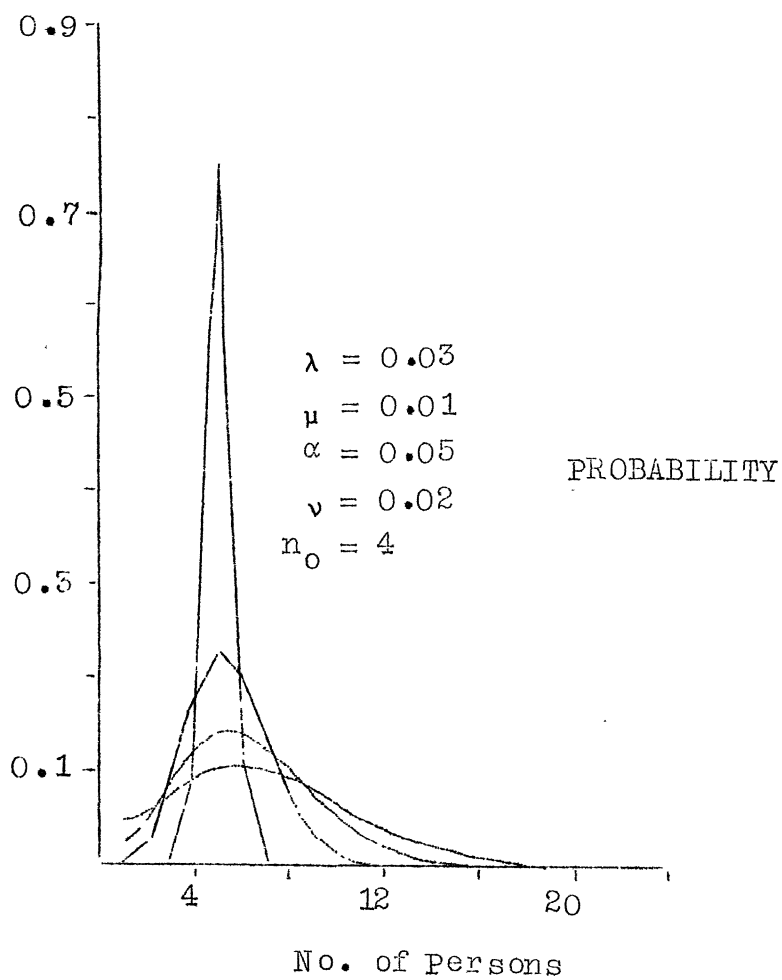

0.9
0.7
0.5
0.3
0.1
λ = 0.03
μ = 0.01
α = 0.05
ν = 0.02
n0 = 4
PROBABILITY
4
12
20
No. of Persons

Fig. 4.2

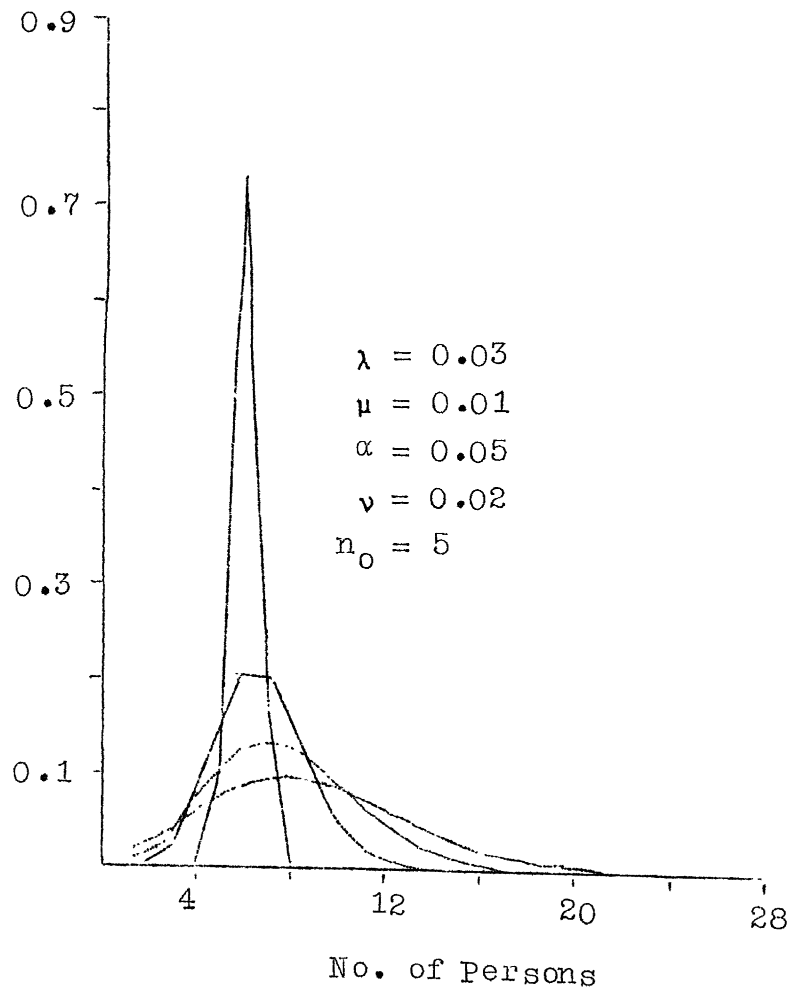

0.9
0.7
0.5
0.3
0.1
λ = 0.03
μ = 0.01
α = 0.05
ν = 0.02
n0 = 5
4
12
20
28
No. of Persons

Fig. 4.3

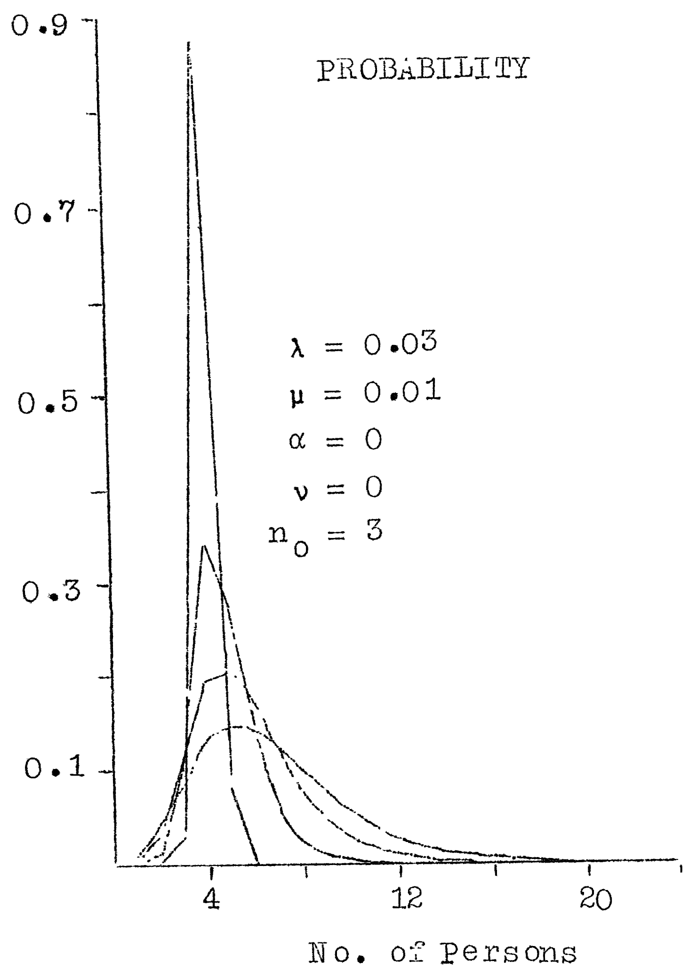

PROBABILITY
0.9
0.7
0.5
0.3
0.1
λ = 0.03
μ = 0.01
α = 0
ν = 0
n_o = 3
4
12
20
No. of Persons

Fig. 4.4

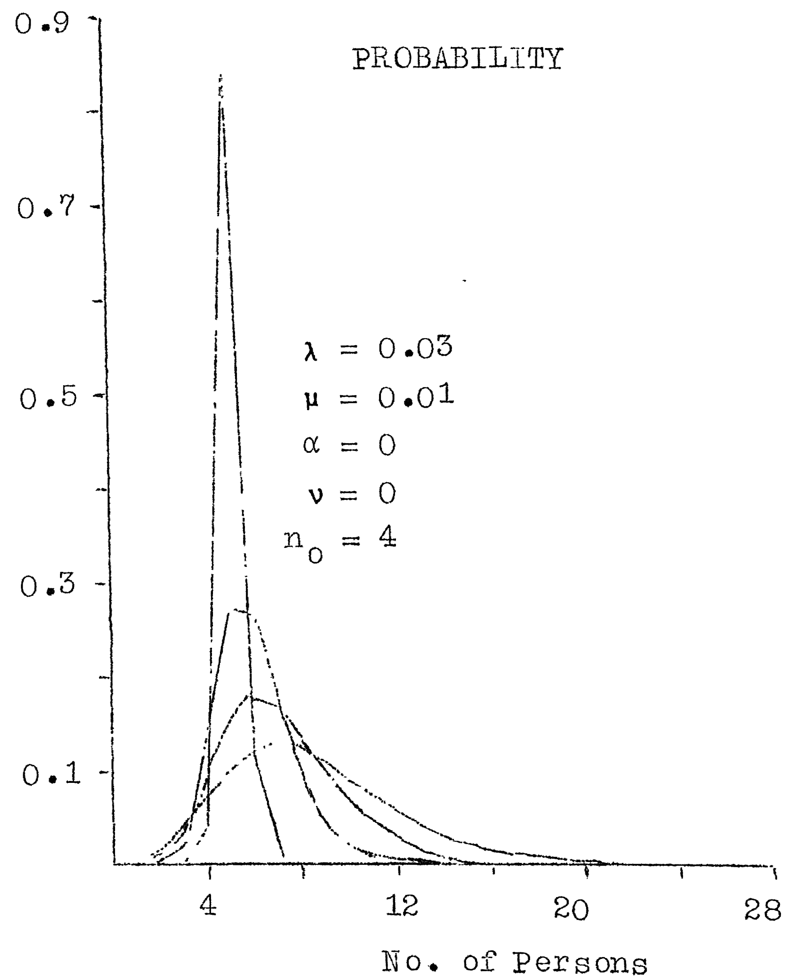

PROBABILITY
0.9
0.7
0.5
0.3
0.1
λ = 0.03
μ = 0.01
α = 0
ν = 0
n_o = 4
4
12
20
28
No. of Persons

Fig. 4.5

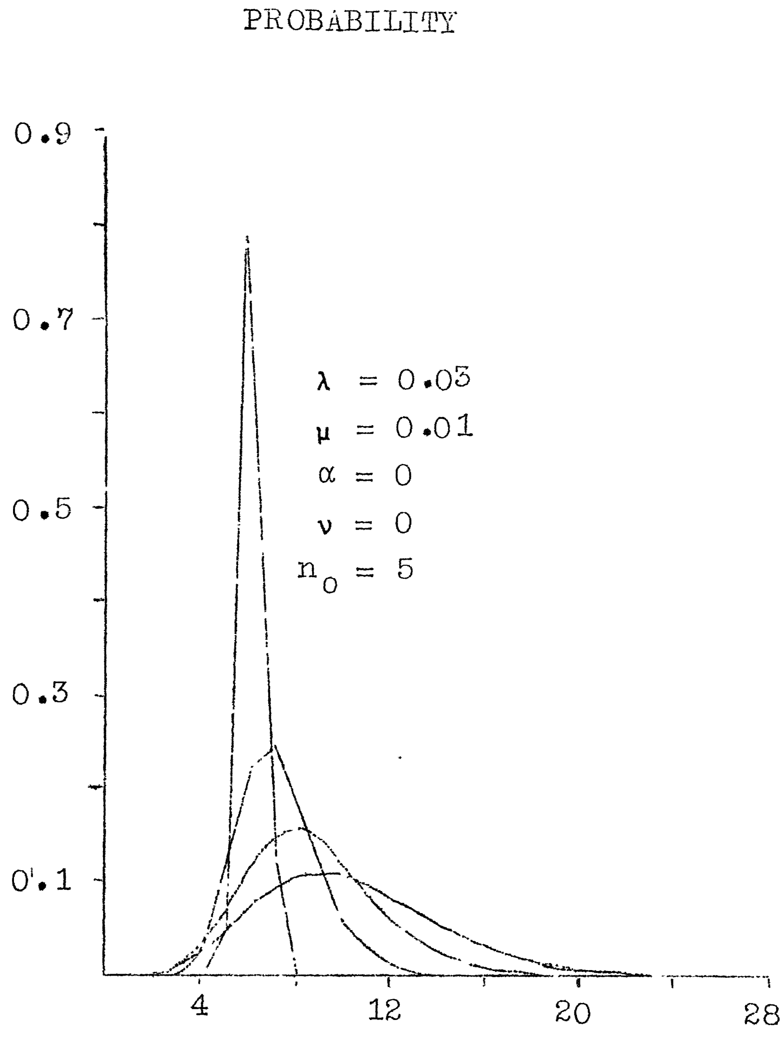

PROBABILITY
0.9
0.7
0.5
0.3
0.1
λ = 0.03
μ = 0.01
α = 0
ν = 0
n_0 = 5
4
12
20
28
No. of Persons

Fig. 4.6

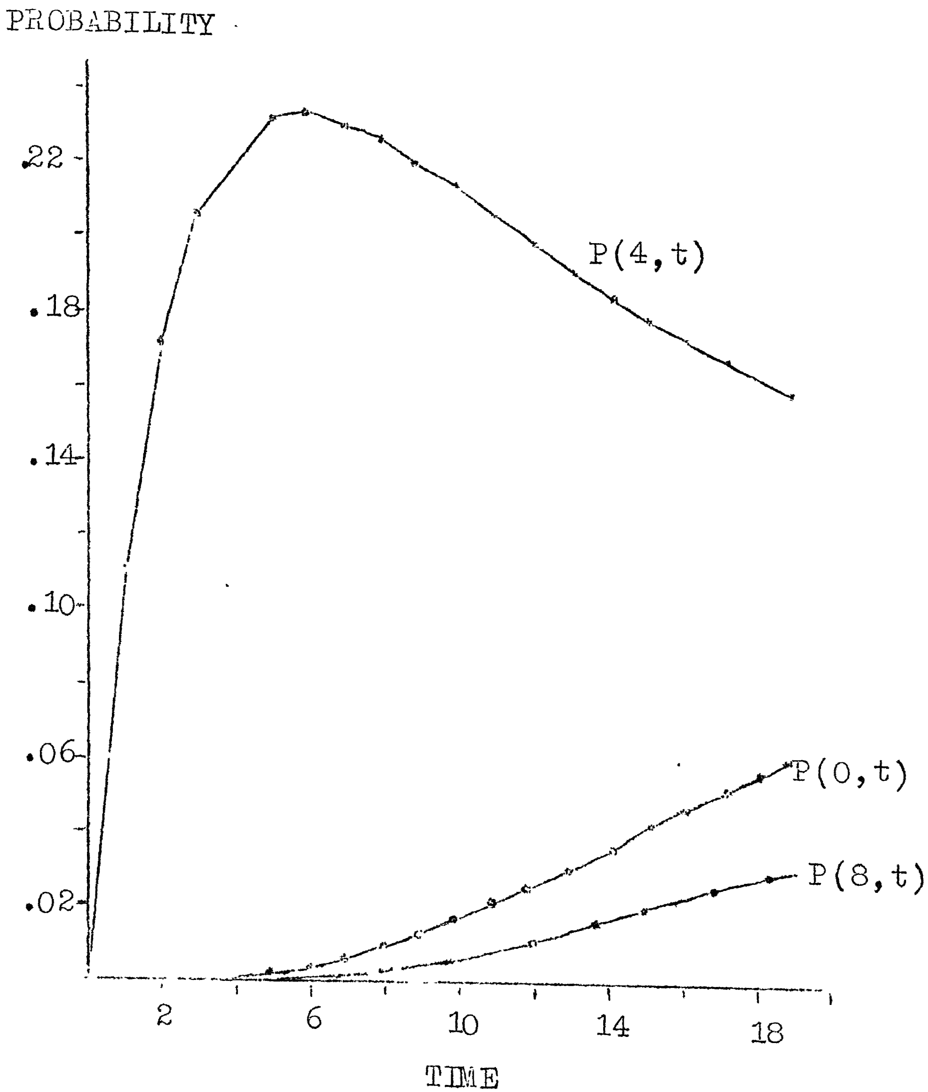

PROBABILITY
.22
.18
.14
.10
.06
.02
P(4,t)
P(0,t)
P(8,t)
2
6
10
14
18
TIME

Fig. 4.7

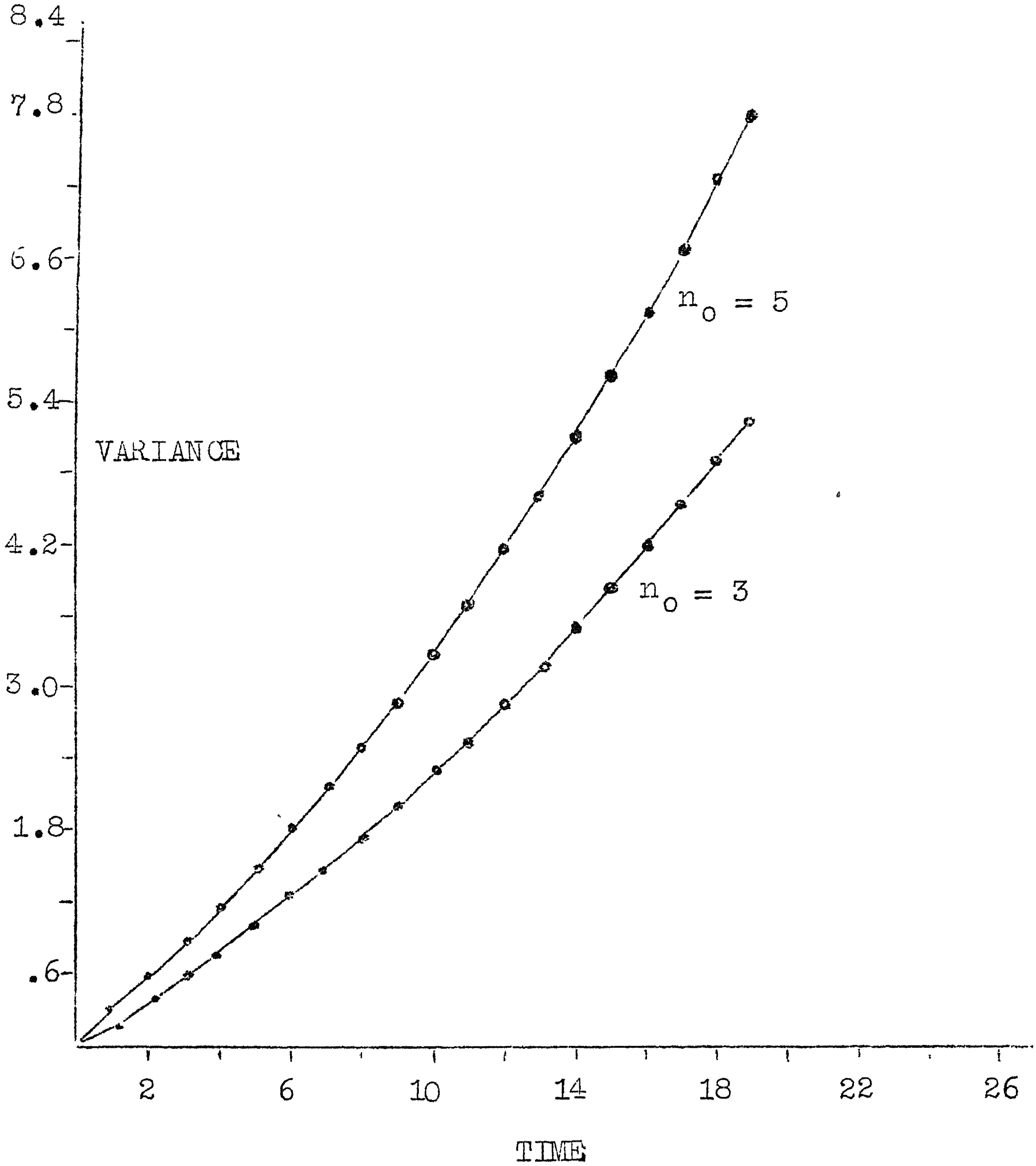


Fig. 4.10

CHAPTER V

ON MULTIPLE INFECTION IN EPIDEMICS

5.1 INTRODUCTION : As we have mentioned in Chapter I, all the work in epidemics whether on the theory development or its applications has been based on the assumption that only single infection take place in the time interval $(t, t + \Delta t)$. In other words, we have assumed that the probability of two or more infections in this interval is an infinitesimal of a higher order than the probability of a single infection in this interval. The case with multiple births in birth and death process has already been considered by Doubleday [1973], Kapur [1979e] and Kapur and Kumar [1978]. We assume here in this chapter that more than one infection can take place in $(t, t + \Delta t)$. Our attempt is to get insight into such situations. To begin with, we consider very simple models in epidemics based on Bailey [1957].

The first model is a simple epidemic model which describes the situation when susceptibles are getting infected with no removal. The second is general epidemic model which takes into account that infectives are being removed either by death, isolation or permanent immunity. Both deterministic and stochastic versions are considered and compared.

5.2 SIMPLE EPIDEMIC MODEL

5.2.1 DETERMINISTIC MODEL : Let x and y denote the number of susceptibles and infectives respectively at any time in the total population size n+1 with $x(0) = n$ and $y(0) = 1$. Let β_1, $2\beta_2$, $3\beta_3, \ldots, \ldots m\beta_m$ denote the infection rates of there being one person infected, two persons infected,..., m persons infected at a time. The number of new infections in time dt is $(\beta_1 + 2\beta_2 + m\beta_m)\, xy\, dt$. The differential equation which describes the process is as follows

$$\frac{dx}{dt} = - \beta_1 xy - 2\beta_2 xy - \ldots - m\beta_m xy \quad (m < n)$$

with $x + y = n + 1$.

Changing the time scale to $\tau = t\beta_1$, we get

$$\frac{dx}{d\tau} = -(1 + 2\epsilon_2 + 3\epsilon_3 + \ldots + m\epsilon_m)\, xy, \tag{1}$$

$$\text{where} \quad \epsilon_i = \frac{\beta_i}{\beta_1} \quad (i = 2,3,\ldots,m) \tag{2}$$

Substituting the value of y, we obtain

$$\frac{dx}{d\tau} = -(1+2\epsilon_2+3\epsilon_3+\ldots+m\epsilon_m)\, x(n+1-x) \tag{3}$$

with initial condition

$$x(0) = n \tag{4}$$

Integrating (3), we get

$$x(\tau) = \frac{n(n+1)}{n + e^{(1+2\epsilon_2+3\epsilon_3+\ldots+m\epsilon_m)(n+1)\tau}} \tag{5}$$

when there is only single infection i.e. $\epsilon_2 = \epsilon_3 = \epsilon_m = 0$ then

$$x(\tau) = \frac{n(n+1)}{n + e^{(n+1)\tau}} \tag{6}$$

We observe that at any time t, the number of susceptibles are reduced since β_1 in the case of single infection in replaced by $\beta_1 + 2\beta_2 + \ldots + m\beta_m$ in the case of multiple infections.

For the epidemic curve, we find out $-\frac{dx}{d\tau}$ to know the rate at which new cases occur

$$-\frac{dx}{d\tau} = \frac{n(n+1)^2(1+2\epsilon_2+3\epsilon_3+\ldots+m\epsilon_m)e^{(1+2\epsilon_2+3\epsilon_3+\ldots+m\epsilon_m)(n+1)\tau}}{\{n + e^{(1+2\epsilon_2+\ldots+m\epsilon_m)(n+1)\tau}\}^2} \tag{7}$$

This curve attains its maximum at $\tau = \frac{\log n}{(1+2\epsilon_2+\ldots+m\epsilon_m)(n+1)}$,

$x = \frac{(n+1)}{2}$ and $-\frac{dx}{d\tau} = \frac{(n+1)^2\ (1+2\epsilon_2+3\epsilon_3+\ldots+m\epsilon_m)}{4}$.

Epidemic curve given by (7) is plotted for $n = 6$, $m = 2$ in figures 5.1, 5.2 for $\epsilon = 0.2$ and 0.5 where it is compared with the corresponding curve obtained for the stochastic case obtained in Section 5.2.3.

5.2.2 STOCHASTIC MODEL : Let n susceptibles and a single infective be there in a population of size n+1. Let $p(r,t)$ denote the probability of there being r susceptibles in the system at any time t. Assuming i infections in $(t, t + \Delta t)$ $(i = 1,2,\ldots,m)$ with $\beta_1, \beta_2, \ldots, \beta_m$ the infection rates, let

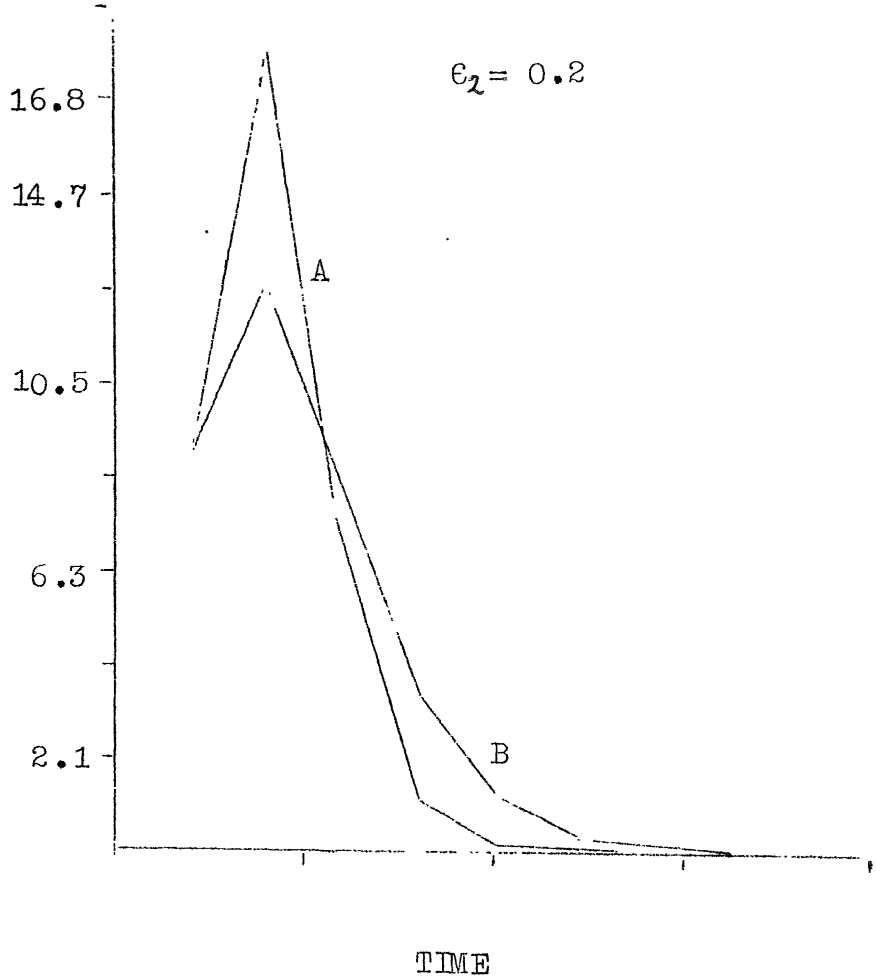


Fig. 5.1 : Comparison of deterministic and stochastic epidemic curves for n = 6.
A - Deterministic B - Stochastic

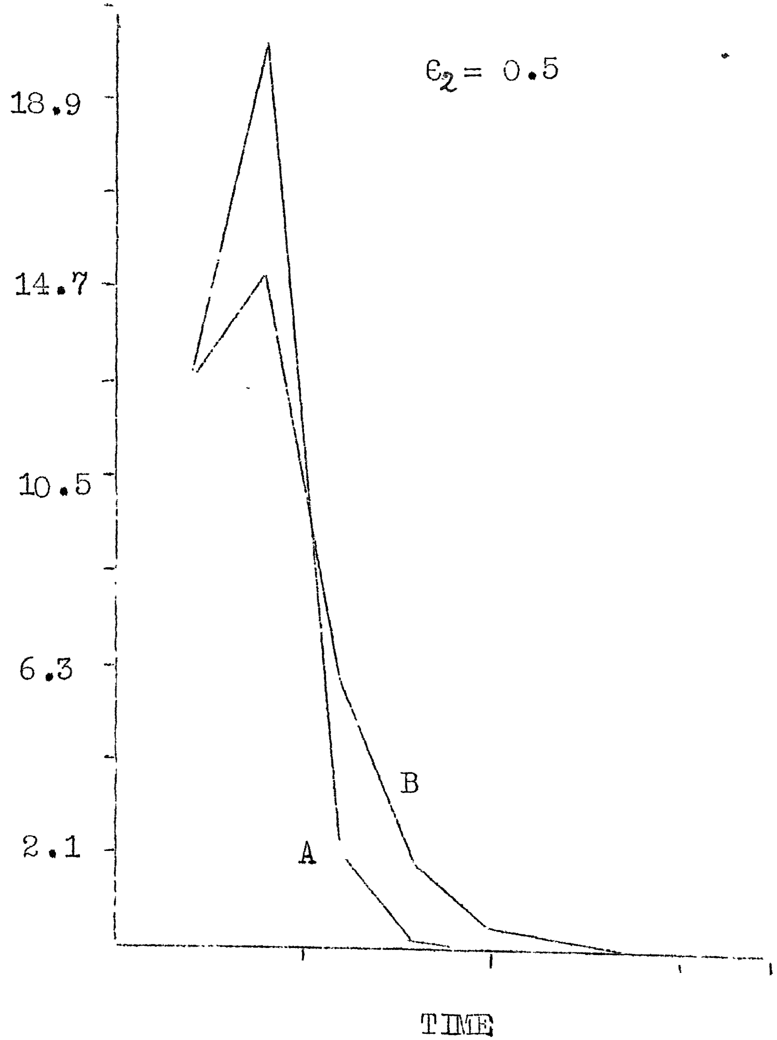


Fig. 5.2 : Comparison of deterministic and stochastic epidemic curves for n = 6.
A - Deterministic B - Stochastic

$\beta_1 r(n+1-r)\,\Delta t$, $\beta_2 r\,(n+1-r)\,\Delta t$, ..., $\beta_m r(n+1-r)\,\Delta t$ be the probabilities of one infection, two infections, three infections, ..., m infections when the number of susceptibles is r at any time. When we write down the relation between probabilities of different states of the system, we are unable to get the differential-difference equations system as compact as in the case of simple epidemic model with single infection. Here the system is very much dependent on m and n. Of course we assume that $m < n$ with m very small in comparison to n.

For the case $m \leq n/2$ when n is even or $m \leq \frac{n-1}{2}$ when n is odd, we get a differential-difference equation system in which the first m equations $(r = 0,1,2,\ldots,m-1)$ and the last m equations $(r = n-m+1,\ldots,n)$ are written separately with great care on the relations between probabilities of different states. But the middle $n+1 - 2m$ equations can be represented by a single equation with r varying from m to n-m. Changing the time-scale to $\tau = t\beta_1$ and writing $\epsilon_i = \frac{\beta_i}{\beta_1}$ $(i = 2,3,\ldots,m)$, we get the following system of differential-difference equations

$$\frac{dp(0,\tau)}{d\tau} = np(1,\tau) + \epsilon_2\, 2(n-1)p(2,\tau)+\ldots+m(n-m+1)\epsilon_m\, p(m,\tau)$$

$$\frac{dp(1,\tau)}{d\tau} = 2(n-1)p(2,\tau)+3(n-2)\epsilon_2\, p(3,\tau)+\ldots+(m+1)(n-m)\epsilon_m\, p(m+1,\tau) - n\,p(1,\tau).$$

- -

- -

- -

$$\frac{dp(m-1,\tau)}{d\tau} = m(n+1-m)p(m,\tau)+(m+1)(n-m)\epsilon_2\, p(m+1,\tau) + \ldots + (2m-1)(n+2-2m)\epsilon_m\, p(2m-1,\tau) - (1+\epsilon_2+\epsilon_3+\ldots+\epsilon_{m-1})(m-1)(n-m+2)\, p(m-1,\tau)$$

$$\frac{dp(r,\tau)}{d\tau} = (r+1)(n-r)p(r+1,\tau)+(r+2)(n-r-1)\epsilon_2\, p(r+2,\tau)$$
$$+ (r+3)(n-r-2)\epsilon_3\, p(r+3,\tau)+\dots+(r+m)(n-r-m+1)\epsilon_m\, p(r+m,\tau)$$
$$-(1+\epsilon_2+\epsilon_3+\dots+\epsilon_m)\, r(n+1-r)\, p(r,\tau)$$
$$r = m, m+1, \dots, n-m$$

$$\frac{dp(n-m+1,\tau)}{d\tau} = (n-m+2)(m-1)\, p(n-m+2,\tau)+\epsilon_2(n-m+3)(m-2)p(n-m+3,\tau)$$
$$+\dots+\epsilon_{m-1}\, np(n,\tau)-(1+\epsilon_2+\epsilon_3+\dots+\epsilon_m)(n-m+1)mp(n-m+1,\tau)$$

- -

- (8)

- -

$$\frac{dp(n-1,\tau)}{d\tau} = np(n,\tau) - (1+\epsilon_2+\epsilon_3+\dots+\epsilon_m)\, 2(n-1)\, p(n-1,\tau)$$

$$\frac{dp(n,\tau)}{d\tau} = -(1+\epsilon_2 + \epsilon_3 + \dots + \epsilon_m)\, np(n,\tau)$$

with initial conditions

$$p(r,0) = 1 \quad \text{when } r = n$$
$$p(r,0) = 0 \quad \text{when } r \neq n. \tag{9}$$

For the case $m > n/2$ when n is even or $m > \frac{n-1}{2}$ when n is odd, we have to write down the $n+1$ equations separately. In this case no two equations can be combined into single equation. Following the previous procedure, we get the differential-difference equation system as follows.

$$\frac{dp(0,\tau)}{d\tau} = np(1,\tau)+2(n-1)\epsilon_2\, p(2,\tau) +\dots+ m(n-m+1)\,\epsilon_m\, p(m,\tau)$$

$$\frac{dp(1,\tau)}{d\tau} = 2(n-1)p(2,\tau)+3(n-2)\epsilon_2\, p(3,\tau)+\dots+\epsilon_m(m+1)(n-m)p(m+1,\tau)$$
$$- np(1,\tau)$$

- -

- -

- -

$$\frac{dp(m-1,\tau)}{d\tau} = m(n+1-m)p(m,\tau)+(m+1)(n-m)\,\epsilon_2\,p(m+1,\tau) + \ldots + \epsilon_{n-m+1}\,np(n,\tau)-(1+\epsilon_2+\epsilon_3+\ldots+\epsilon_{m-1})(m-1)(n-m+2)p(m-1,\tau)$$

$$\frac{dp(m,\tau)}{d\tau} = (m+1)(n-m)\,p(m+1,\tau)+(m+2)(n-m-1)\,\epsilon_2\,p(m+2,\tau)+\ldots + \epsilon_{n-m}\,np(n,\tau)-(1+\epsilon_2+\epsilon_3+\ldots+\epsilon_m)m(n-m+1)\,p(m,\tau)$$

$$\frac{dp(m+1,\tau)}{d\tau} = (m+2)(n-m-1)p(m+2,\tau)+(m+3)(n-m-2)\epsilon_2\,p(m+3,\tau)+\ldots +\epsilon_{n-m-1}\,np(n,\tau)-(1+\epsilon_2+\epsilon_3+\ldots+\epsilon_m)(m+1)(n-m)p(m+1,\tau)$$

- (9)

- -

- -

$$\frac{dp(n-1,\tau)}{d\tau} = np(n,\tau) - (1+\epsilon_2+\epsilon_3+\ldots+\epsilon_m)\,2(n-1)\,p(n-1,\tau)$$

$$\frac{dp(n,\tau)}{d\tau} = -(1+\epsilon_2+\epsilon_3+\ldots+\epsilon_m)\,np(n,\tau)$$

with initial conditions $p(r,0) = 1$ when $r = n$ otherwise 0

(10)

5.2.3 <u>TWO SPECIAL CASES</u> :

(i) $n = 3,\ m = 2$ i.e. $m < n$ and $m > \frac{n-1}{2}$

$$\frac{dp(0,\tau)}{d\tau} = 3p(1,\tau) + 4\epsilon_2\,p(2,\tau)$$

$$\frac{dp(1,\tau)}{d\tau} = 4p(2,\tau) + 3\epsilon_2\,p(3,\tau) - 3p(1,\tau) \tag{11}$$

$$\frac{dp(2,\tau)}{d\tau} = 3p(3,\tau) - 4(1+\epsilon_2)\,p(2,\tau)$$

$$\frac{dp(3,\tau)}{d\tau} = -3(1+\epsilon_2)\,p(3,\tau)$$

$$p(3,0) = 1;\ p(r,0) = 0\ \ r \neq 3 \tag{12}$$

(ii) $n = 6$, $m = 2$ i.e. $m < n/2$

$$\frac{dp(0,\tau)}{d\tau} = 6p(1,\tau) + 10\epsilon_2\, p(2,\tau)$$

$$\frac{dp(1,\tau)}{d\tau} = 10p(2,\tau) + 12\epsilon_2\, p(3,\tau) - 6p(1,\tau)$$

$$\frac{dp(2,\tau)}{d\tau} = 12p(3,\tau) + 12\epsilon_2\, p(4,\tau) - 10(1+\epsilon_2)\, p(2,\tau)$$

$$\frac{dp(3,\tau)}{d\tau} = 12p(4,\tau) + 10\epsilon_2\, p(5,\tau) - 12(1+\epsilon_2)\, p(3,\tau)$$

$$\frac{dp(4,\tau)}{d\tau} = 10p(5,\tau) + 6\epsilon_2\, p(6,\tau) - 12(1+\epsilon_2)\, p(4,\tau)$$

$$\frac{dp(5,\tau)}{d\tau} = 6p(6,\tau) - 10(1+\epsilon_2)\, p(5,\tau)$$

$$\frac{dp(6,\tau)}{d\tau} = -6(1+\epsilon_2)\, p(6,\tau)$$

$$p(6,0) = 1 \text{ otherwise } 0. \qquad (14)$$

For any value of n and m given, we a get a system of differential-difference equations of first order. This system with initial conditions can be solved numerically by Gill Method. Applying Gill-Method, we solve the two special cases numerically.

5.2.4 GRAPHS

For $n = 6$, $m = 2$

(i) In Figures 5.3 - 5.9, we have plotted $p(0,\tau)$, $p(1,\tau)$, $p(2,\tau)$, $p(3,\tau)$, $p(4,\tau)$, $p(5,\tau)$ and $p(6,\tau)$ respectively against time for $\epsilon_2 = 0$, 0.2 and 0.5.

(ii) In Fig. 5.10, we have plotted all probabilities against time for $\epsilon_2 = 0$.

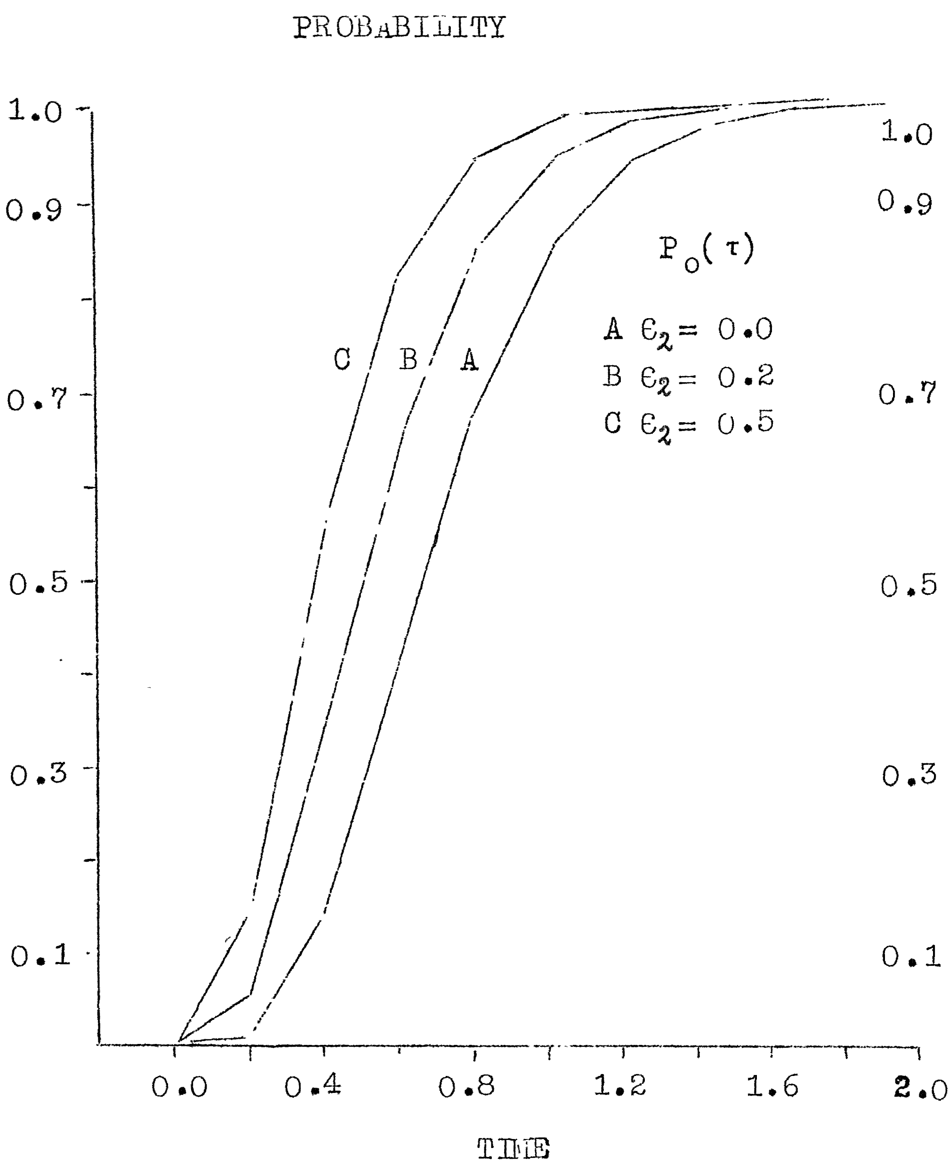

PROBABILITY
1.0
0.9
0.7
0.5
0.3
0.1
0.0
0.4
0.8
1.2
1.6
2.0
TIME
C
B
A
$P_o(\tau)$
A $\epsilon_2 = 0.0$
B $\epsilon_2 = 0.2$
C $\epsilon_2 = 0.5$

Fig. 5.3

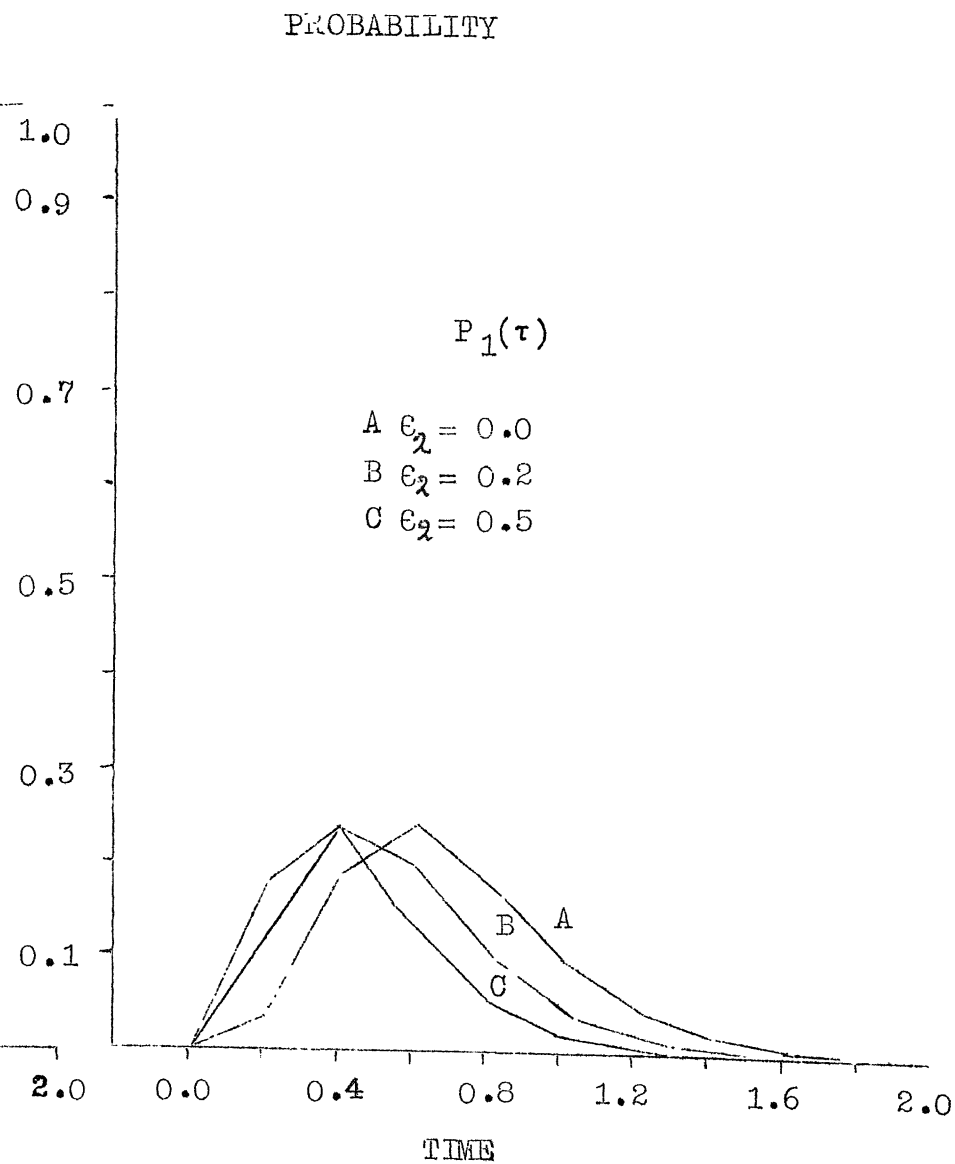

PROBABILITY
1.0
0.9
0.7
0.5
0.3
0.1
0.0
0.4
0.8
1.2
1.6
2.0
TIME
$P_1(\tau)$
A $\epsilon_2 = 0.0$
B $\epsilon_2 = 0.2$
C $\epsilon_2 = 0.5$
A
B
C

Fig. 5.4

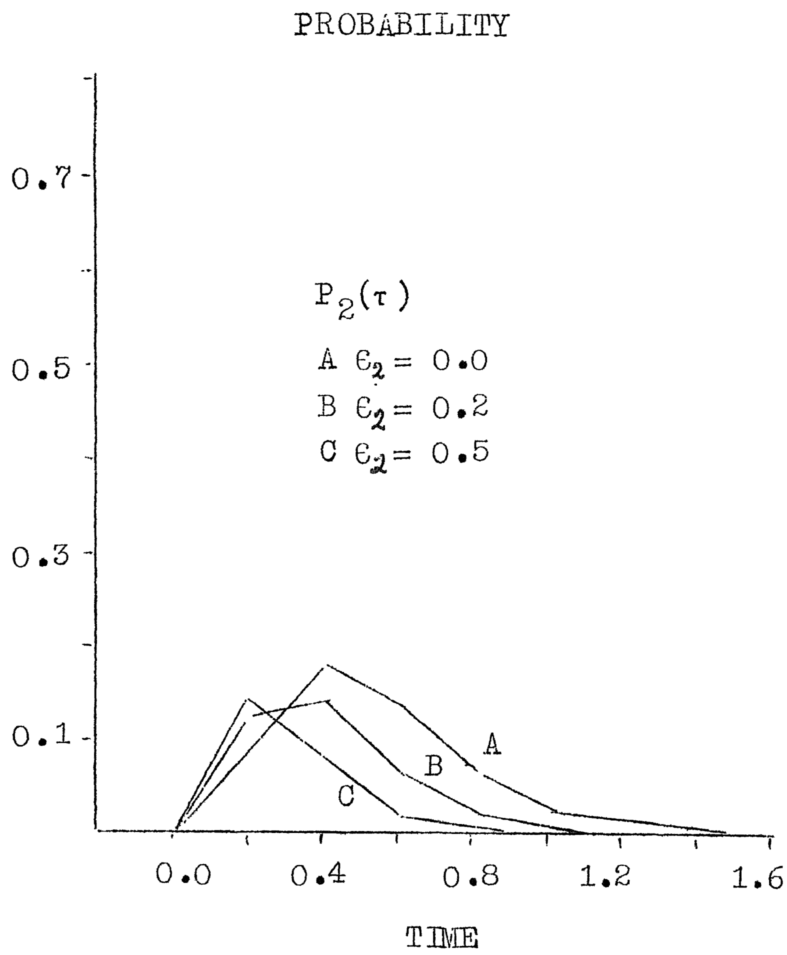

PROBABILITY
0.7
0.5
0.3
0.1
P2(τ)
A ε2 = 0.0
B ε2 = 0.2
C ε2 = 0.5
A
B
C
0.0
0.4
0.8
1.2
1.6
TIME

Fig. 5.5

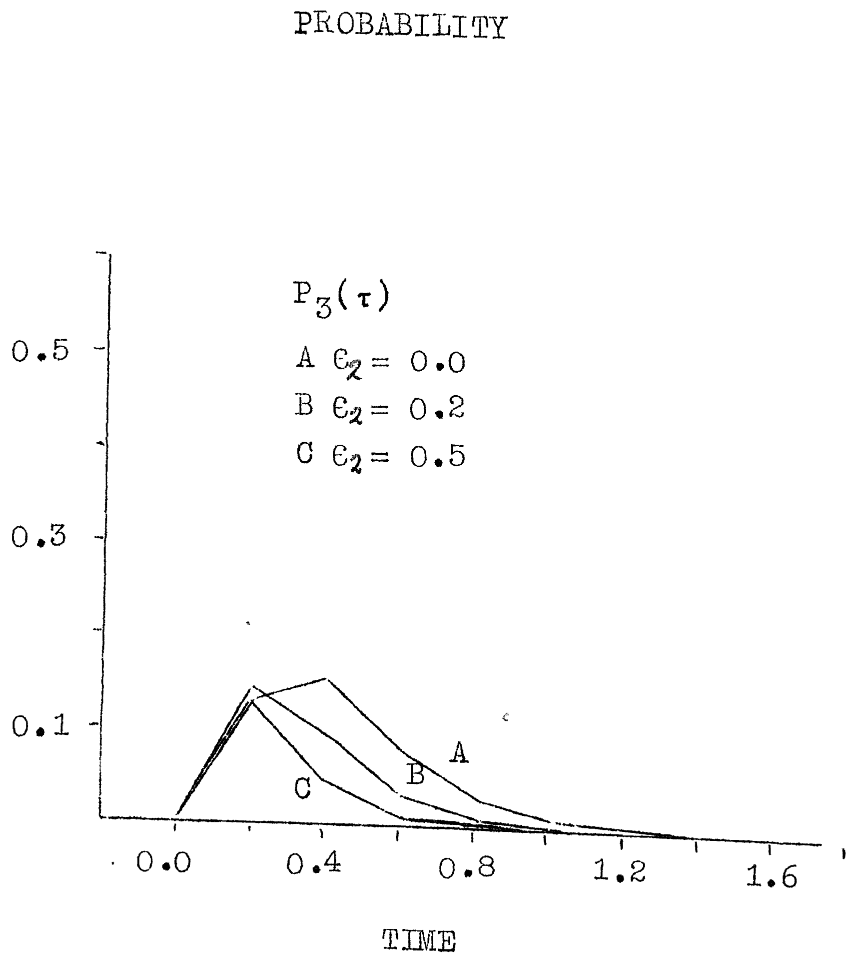
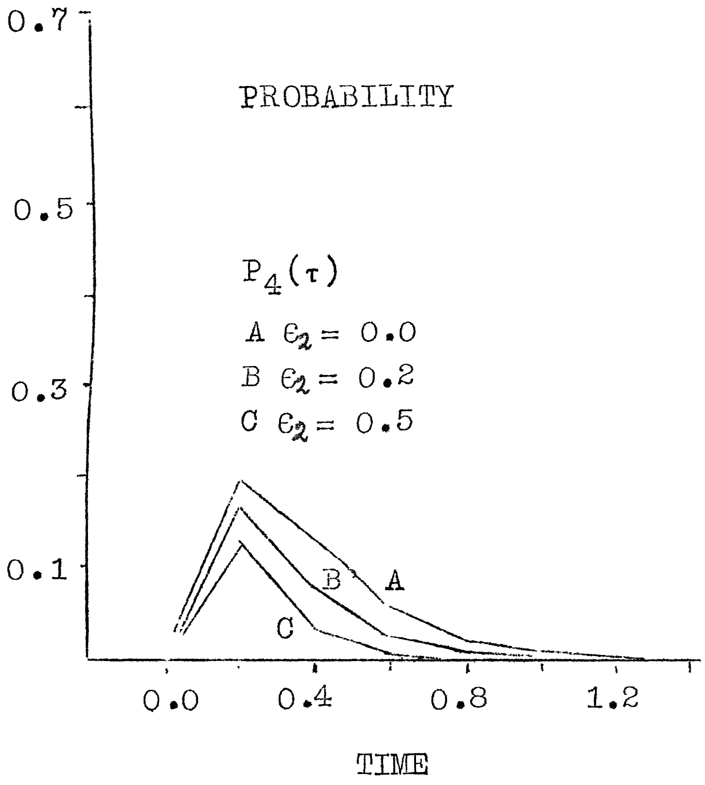
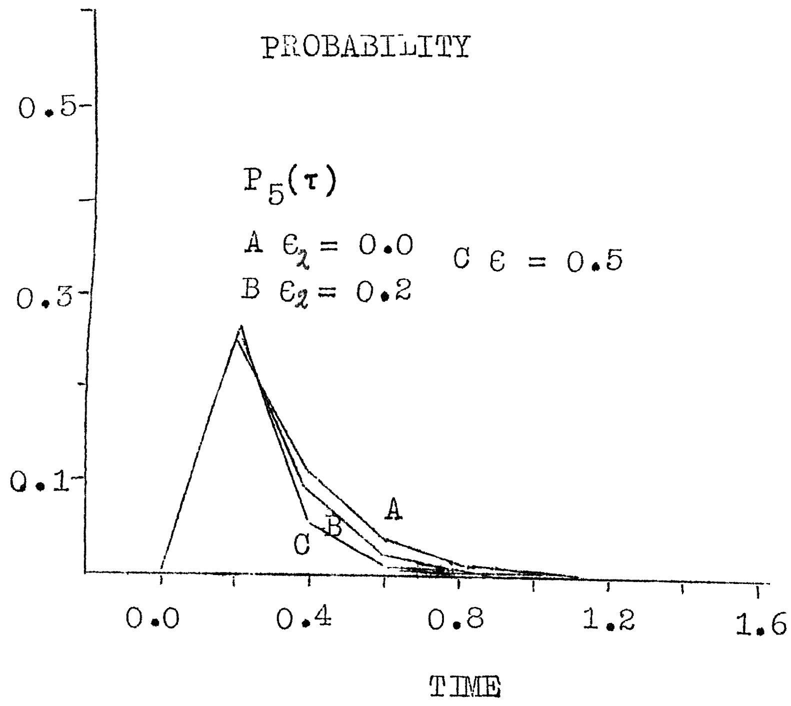

PROBABILITY
0.5
0.3
0.1
P3(τ)
A ε2 = 0.0
B ε2 = 0.2
C ε2 = 0.5
A
B
C
0.0
0.4
0.8
1.2
1.6
TIME

Fig. 5.6



PROBABILITY
$P_4(\tau)$
A $\epsilon_2 = 0.0$
B $\epsilon_2 = 0.2$
C $\epsilon_2 = 0.5$
0.7
0.5
0.3
0.1
0.0
0.4
0.8
1.2
A
B
C
TIME

Fig. 5.7



PROBABILITY
$P_5(\tau)$
A $\epsilon_2 = 0.0$
B $\epsilon_2 = 0.2$
C $\epsilon = 0.5$
0.5
0.3
0.1
0.0
0.4
0.8
1.2
1.6
A
B
C
TIME

Fig. 5.8

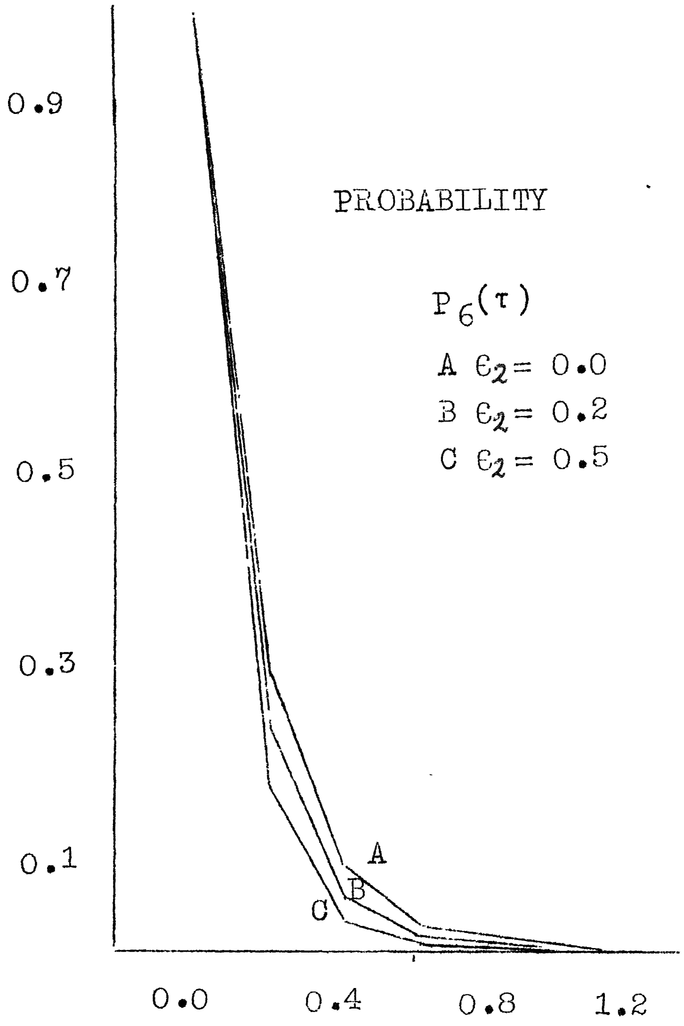


Fig. 5.9

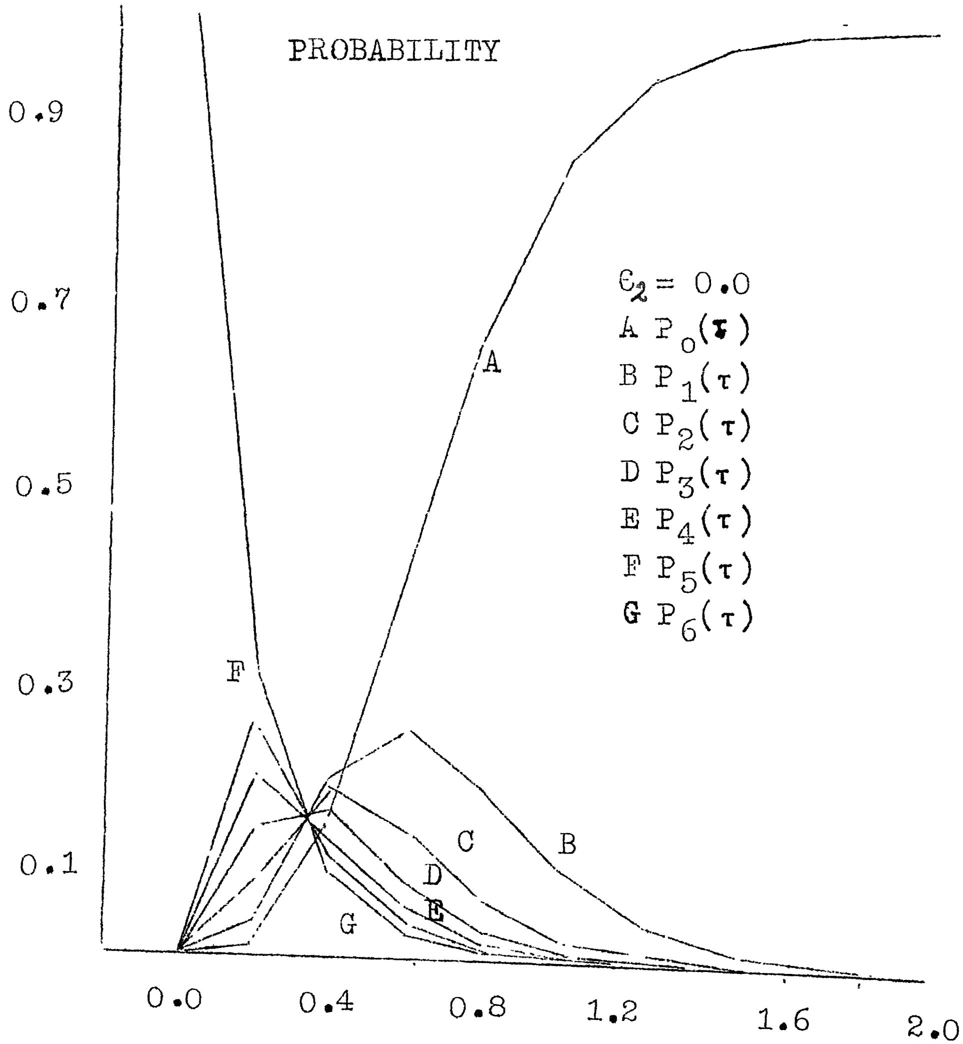


Fig. 5.10

(iii) In Fig. 5.11, we have plotted all probabilities against time for $\epsilon_2 = 0.2$.

(iv) In Fig. 5.12, we have plotted all probabilities against time for $\epsilon_2 = 0.5$.

(v) In Fig. 5.13, we have plotted mean against time for $\epsilon_2 = 0$, 0.2 and 0.5.

(vi) In Fig. 5.14, we have plotted variance against time for $\epsilon_2 = 0$, 0.2 and 0.5.

For $n = 3$, $m = 2$

(i) In Figures 5.15 - 5.18, we have plotted $p(0,\tau)$, $p(1,\tau)$, $p(2,\tau)$ and $p(3,\tau)$ respectively against time for $\epsilon_2 = 0$, 0.2 and 0.5.

(ii) In Fig. 5.19, we have plotted all probabilities against time for $\epsilon_2 = 0$.

(iii) In Fig. 5.20, we have plotted all probabilities against time for $\epsilon_2 = 0.2$.

(iv) In Fig. 5.21, we have plotted all probabilities against time for $\epsilon_2 = 0.5$.

(v) In Fig. 5.22, we have plotted mean against time for $\epsilon_2 = 0$, 0.2 and 0.5.

(vi) In Fig. 5.23, we have plotted variance against time for $\epsilon_2 = 0$, 0.2 and 0.5.

We observe, as expected that multiple infection causes the process of infection to be faster and it takes little time for all the susceptibles to be infected. The probabilities for

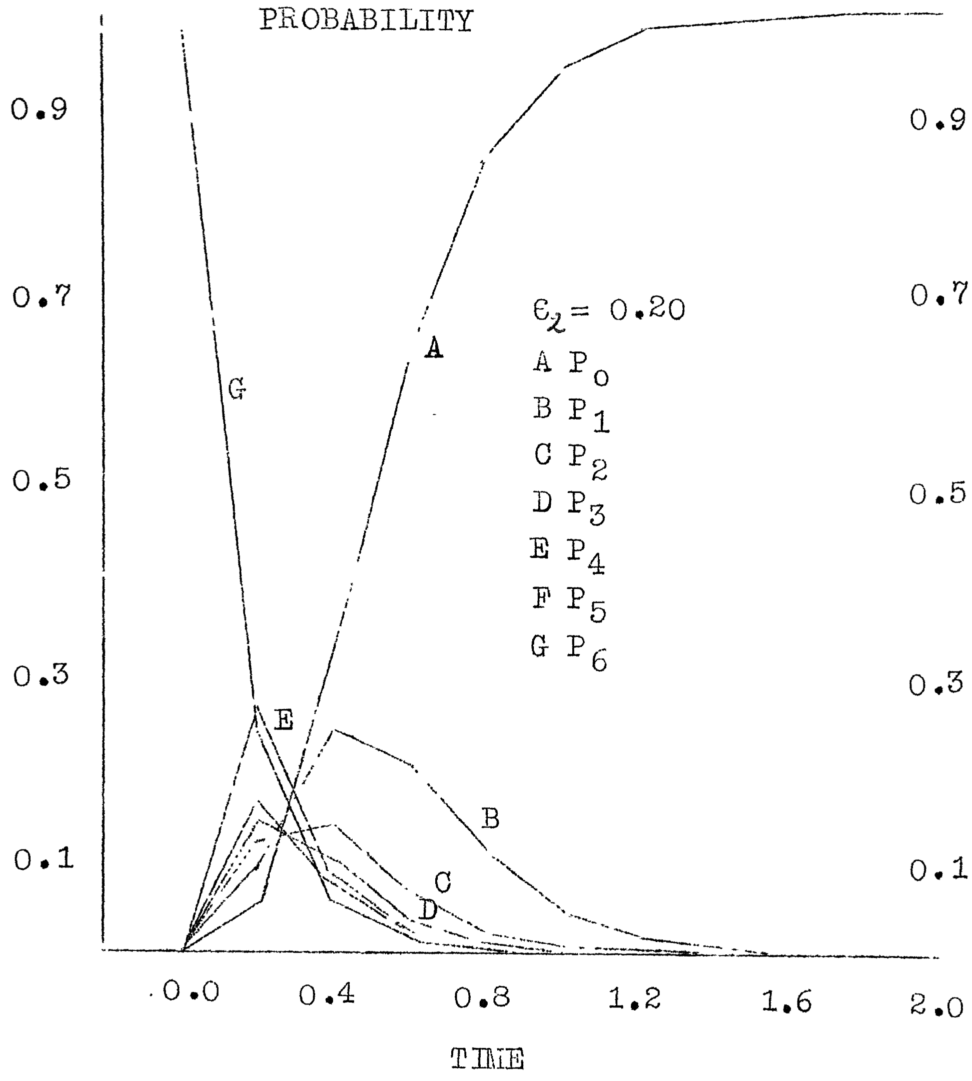


Fig. 5.11

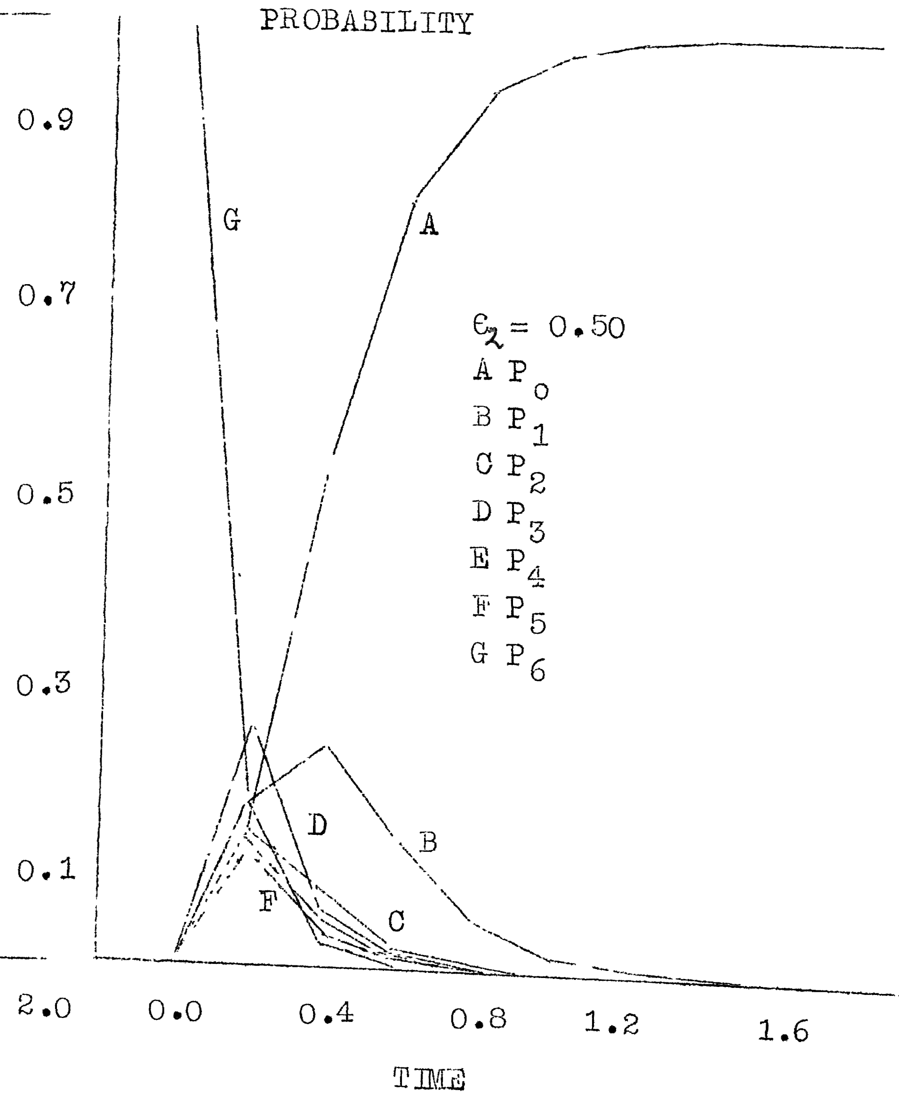


Fig. 5.12

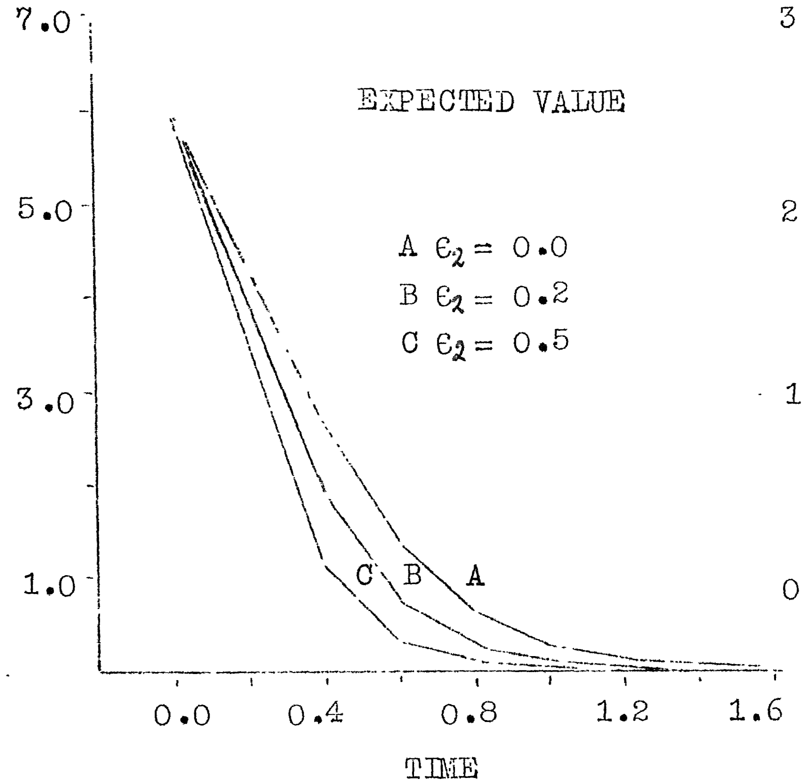


Fig. 5.13

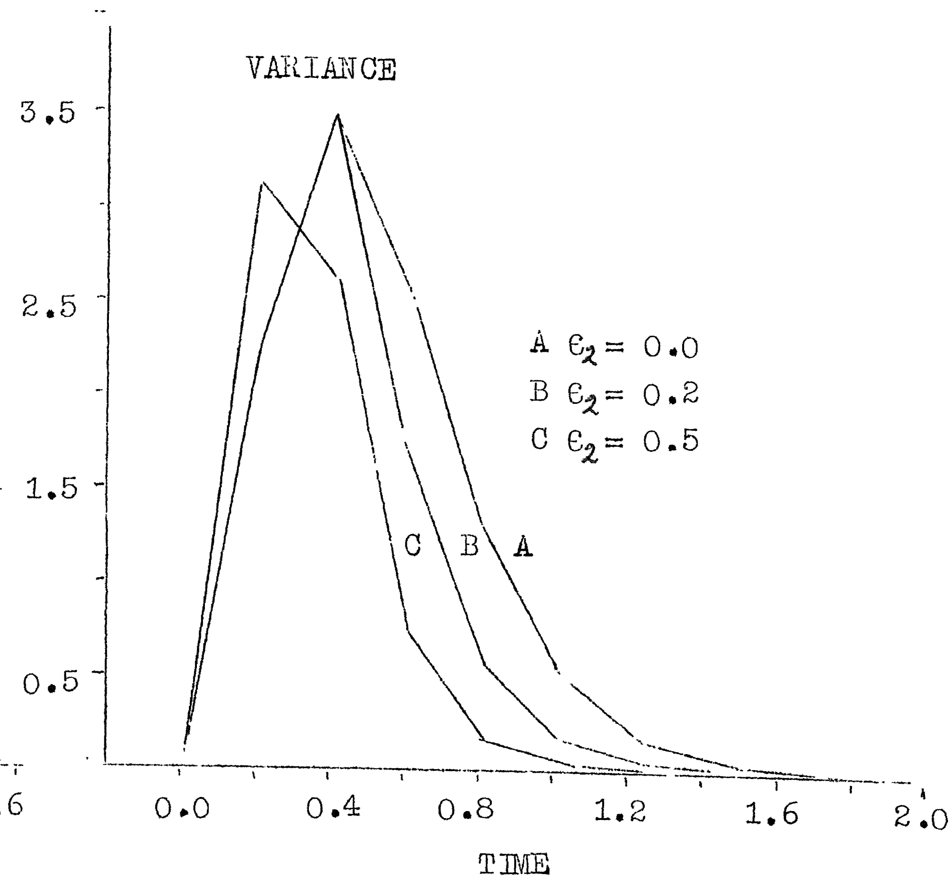


Fig. 5.14

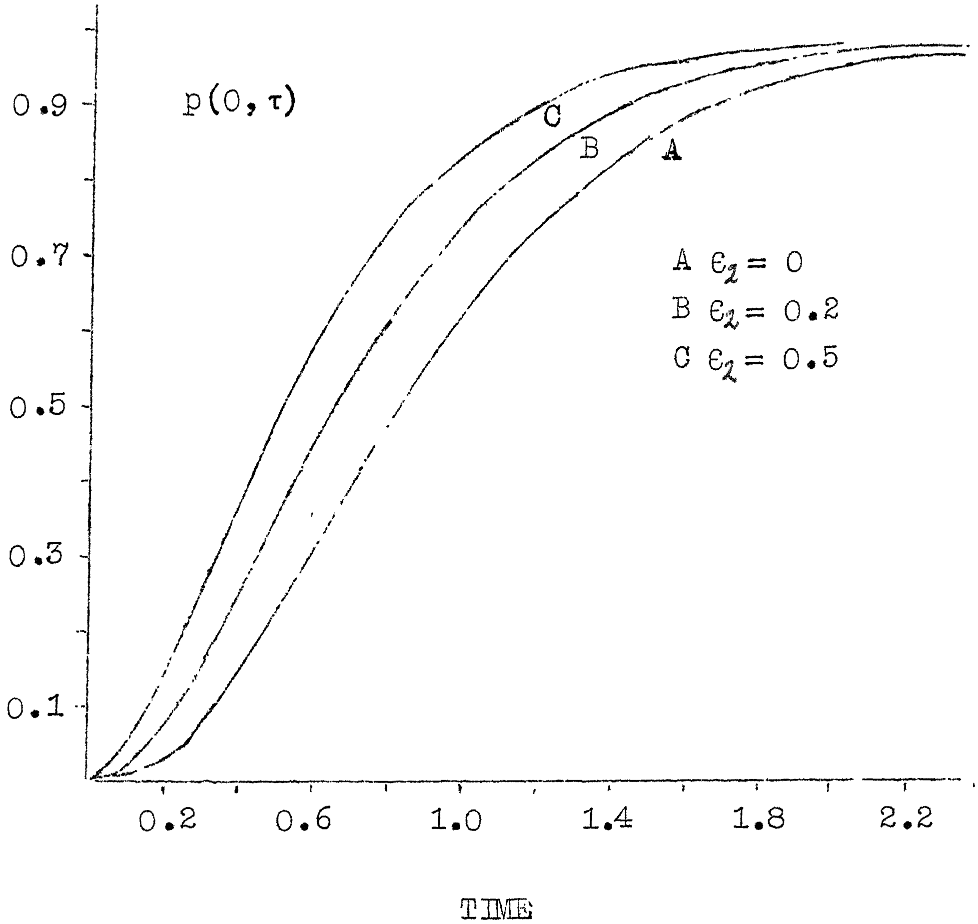


Fig. 5.15

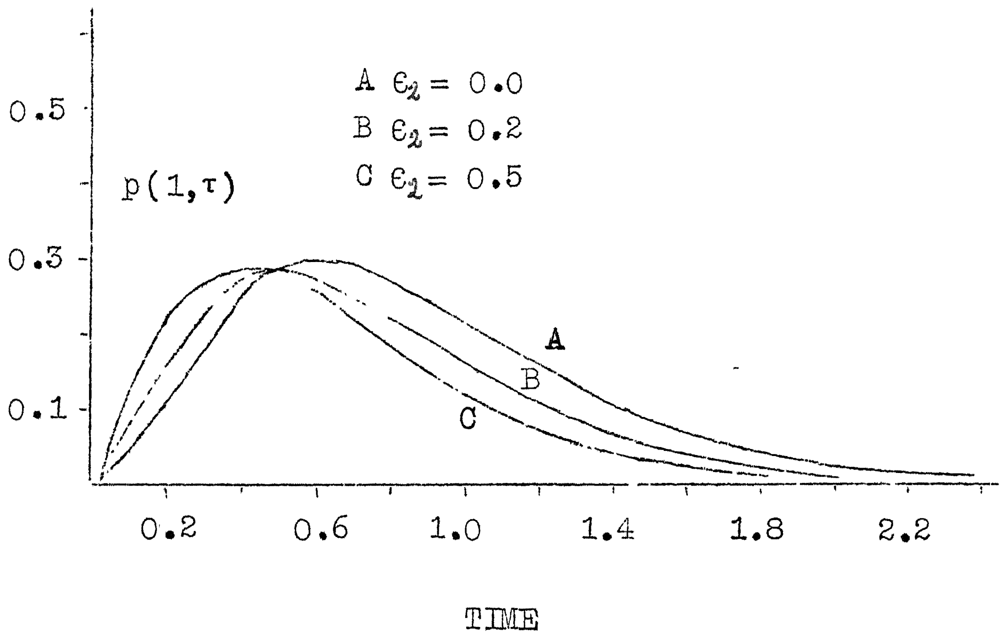


Fig. 5.16

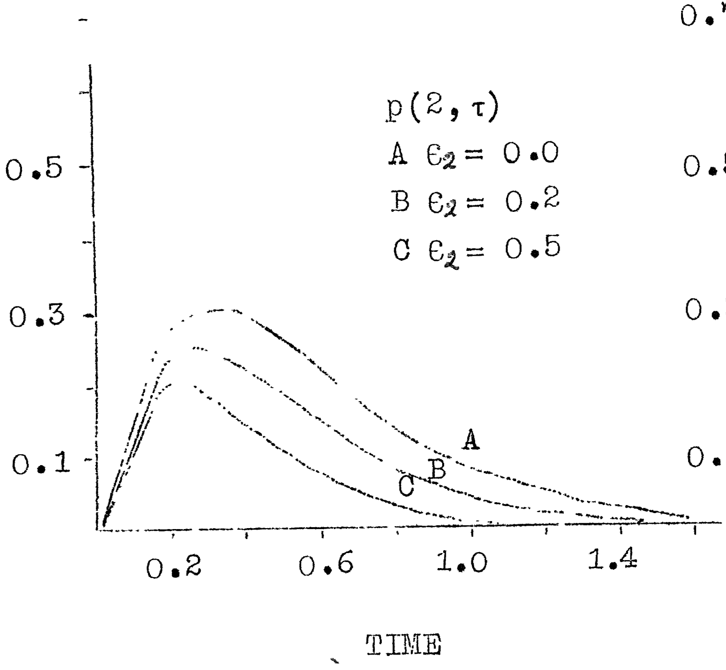


Fig. 5.17

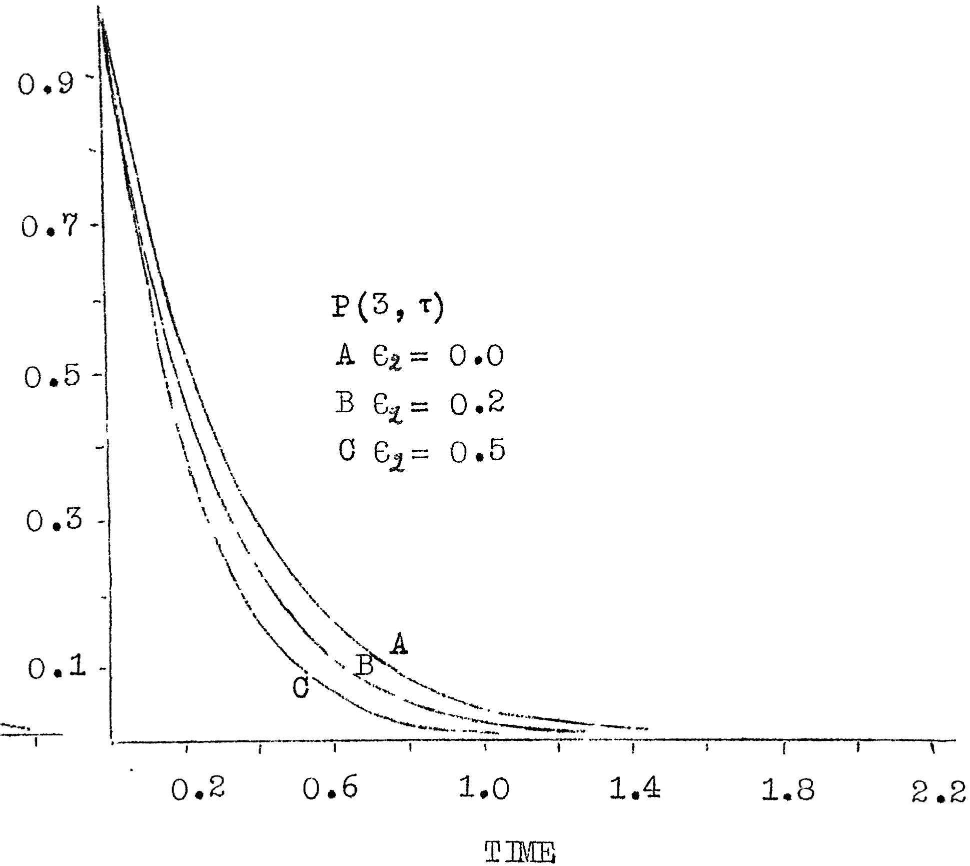


Fig. 5.18

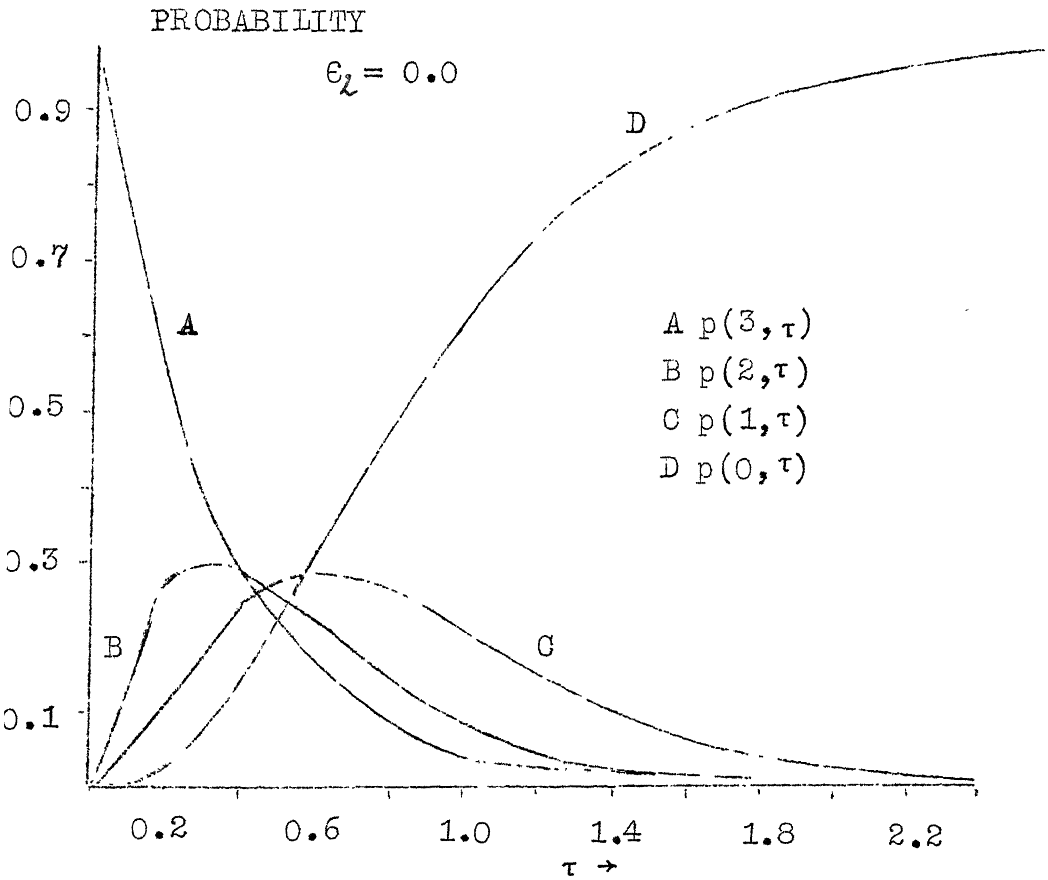


Fig. 5.19

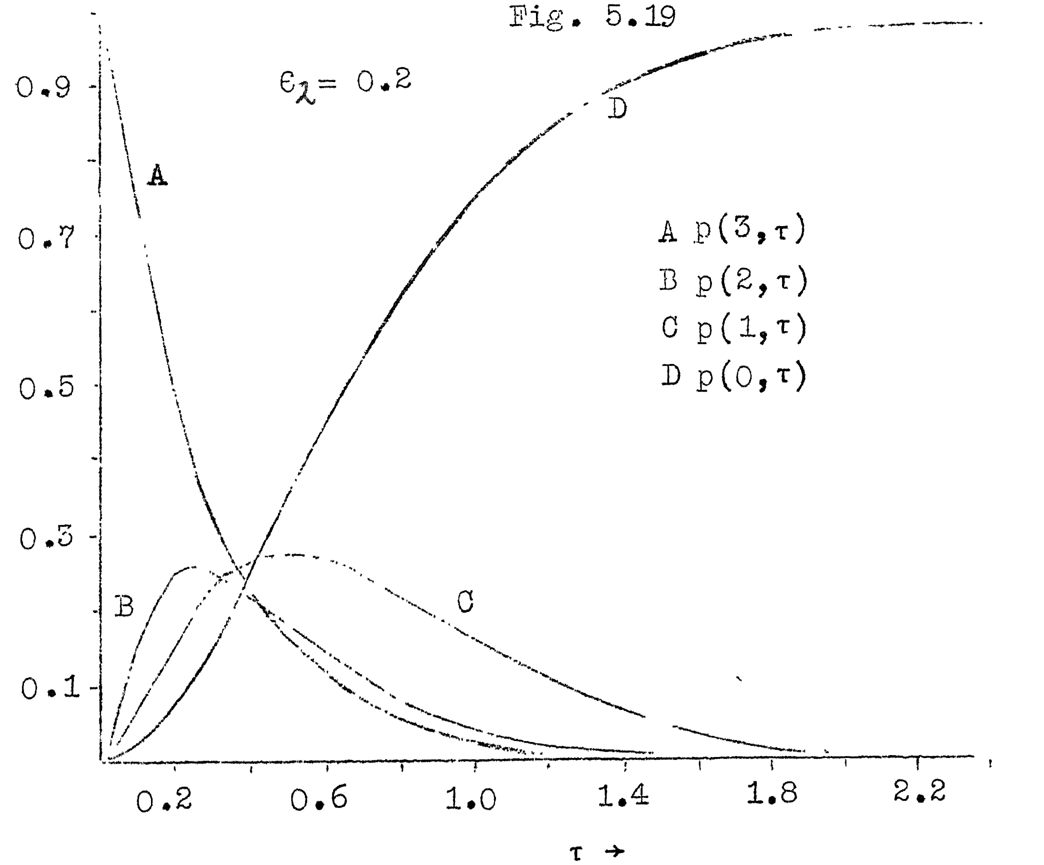


Fig. 5.20

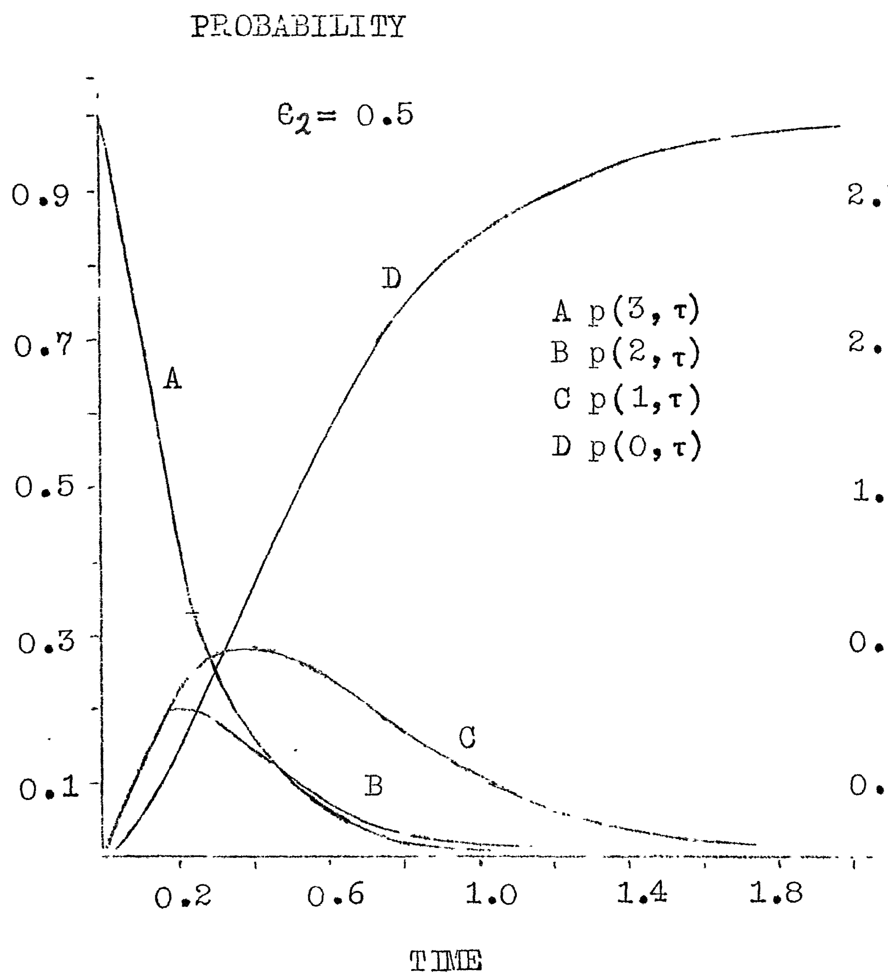


Fig. 5.21

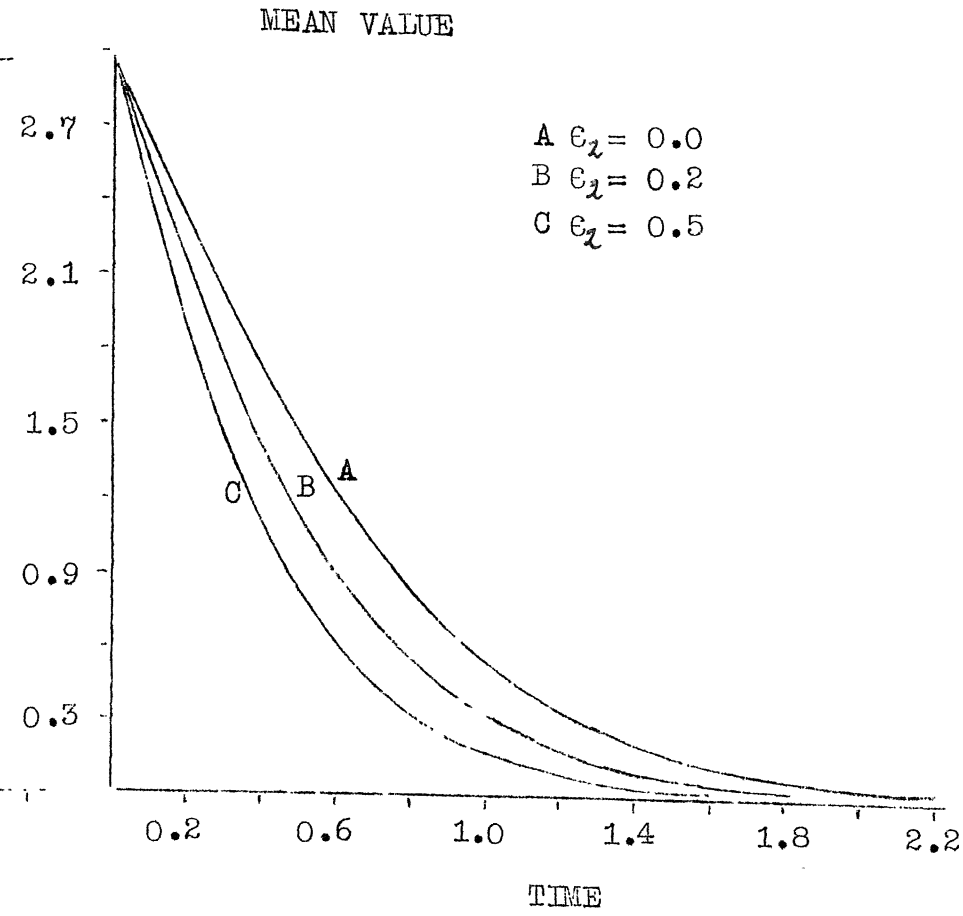


Fig. 5.22

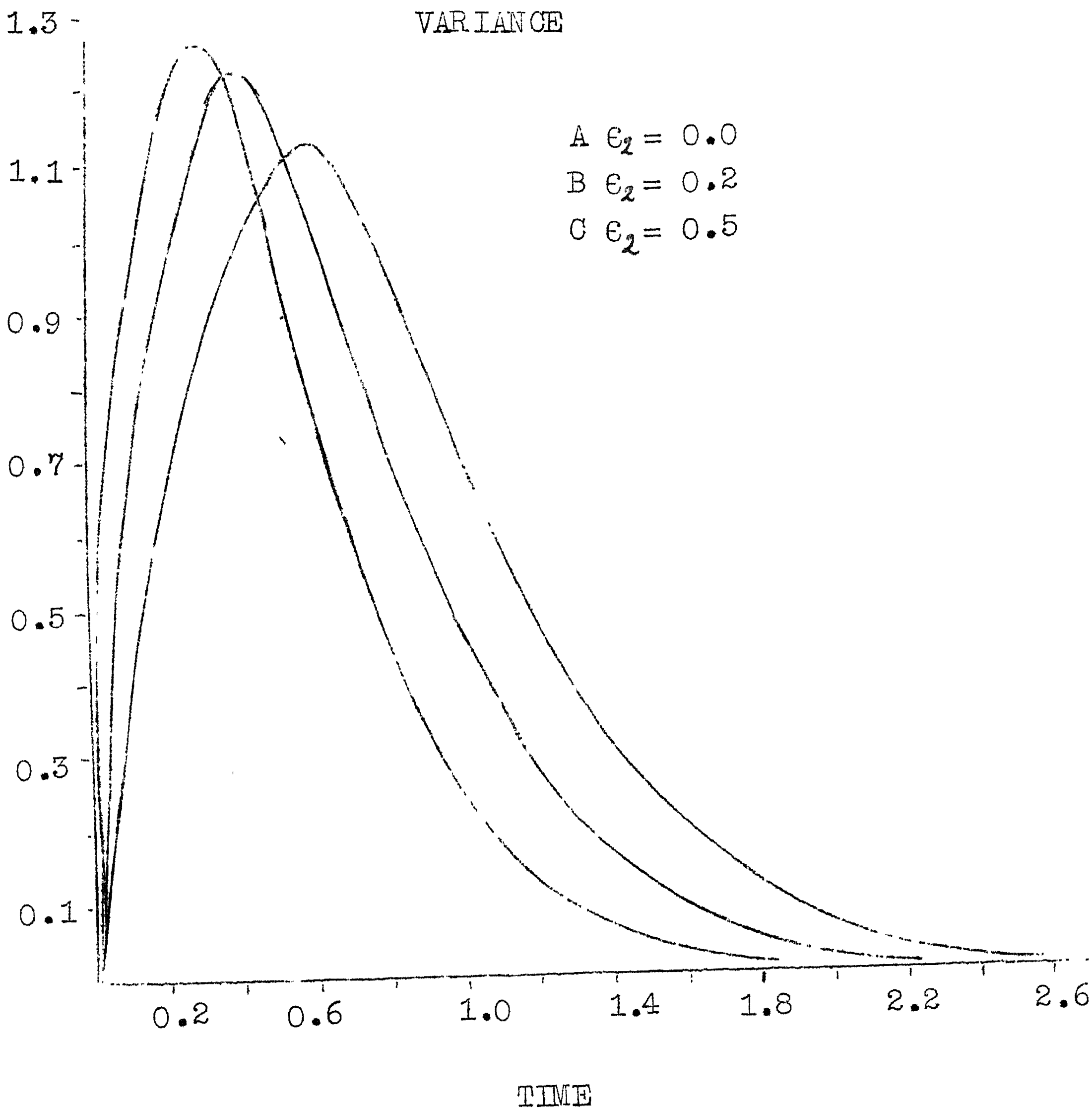

VARIANCE
A $\epsilon_2 = 0.0$
B $\epsilon_2 = 0.2$
C $\epsilon_2 = 0.5$
1.3
1.1
0.9
0.7
0.5
0.3
0.1
0.2
0.6
1.0
1.4
1.8
2.2
2.6
TIME

Fig. 5.23

which r is near to n are being zero faster and for which r is close to zero are reaching 1 fastly.

5.3 GENERAL EPIDEMIC MODEL

5.3.1 STOCHASTIC MODEL

In the population size of n+a where n and a denote the initial number of susceptibles and infectives respectively, we assume i infections ($i = 1,2,\ldots,m$) and single removal in the time interval $(t, t+\Delta t)$ with $\beta_1, \beta_2, \ldots, \beta_m$ as the infection rates and γ the removal rate. Let $p_{rs}(t)$ denote the probability of there being r susceptibles and s infectives in the system at any time t. Let $\beta_i\, rs\, \Delta t$ ($i = 1,2,\ldots,m$) denote the probability of i infections and $\gamma s\, \Delta t$, the probability of single removal in time interval $(t, t+\Delta t)$. Changing the time-scale to $\tau = \beta_1 t$ and writing $\frac{\gamma}{\beta_1} = \rho$ and $\epsilon_i = \frac{\beta_i}{\beta_1}$ ($i=2,3,\ldots,m$), we get the following system of differential-differential equations

$$\frac{dp_{rs}(\tau)}{d\tau} = (r+1)(s-1)\, p_{r+1,s-1}(\tau) + \epsilon_2(r+2)(s-2)\, p_{r+2,s-2}(\tau)$$

$$+\ldots+\epsilon_m(r+m)(s-m)p_{r+m,s-m}(\tau) + p_{r,s+1}(\tau)\quad \rho(s+1)$$

$$-\{(1+\epsilon_2+\epsilon_3+\ldots+\epsilon_{\min(r,m)})\, rs + \rho s\}\quad p_{rs}(\tau) \tag{1}$$

$$r = 0,1,2,\ldots,n-m$$

$$s = m,\ m+1,\ldots,m+(a-1)$$

with initial conditions

$$p_{na}(0) = 1, \quad \text{otherwise } 0 \tag{2}$$

and $\quad 0 \leq r \leq n,\ 0 \leq s \leq n+a,\ 0 \leq r+s \leq n+a \qquad (3)$

With the above restrictions (3) on r, s and r+s, we get $(n+1)(n+a+1) - \frac{n(n+1)}{2}$ number of non-zero probabilities. Of which $a(n-m+1)$ number of differential-difference equations can be obtained directly from the above equation (1), substituting restricted values of r and s.

Even all other probabilities can be obtained from (1) if we make the following convention.

(i) Probability with suffix negative will be taken as zero.

(ii) Condition (3) will be taken care of.

5.3.2 SPECIAL CASE :

We present here as an example a complete system of differential-difference equations for n = 3, a = 2 and m = 2. In this case we get a system of 18 differential-difference equations as follows.

$$\frac{dp_{oo}}{d\tau} = \rho p_{o1}$$

$$\frac{dp_{o1}}{d\tau} = -\rho p_{o1} + 2\rho p_{o2}$$

$$\frac{dp_{o2}}{d\tau} = -2\rho p_{o2} + p_{11} + 3\rho p_{o3}$$

$$\frac{dp_{o3}}{d\tau} = -3\rho p_{o3} + 2\rho p_{12} + 2\epsilon_2 p_{21} + 4\rho p_{o4}$$

$$\frac{dp_{o4}}{d\tau} = -4\rho p_{o4} + 3p_{13} + 4\epsilon_2 p_{22} + 5\rho p_{o5}$$

$$\frac{dp_{05}}{d\tau} = -5\rho\, p_{05} + 4p_{14} + 6\epsilon_2\, p_{23}$$

$$\frac{dp_{10}}{d\tau} = \rho\, p_{11}$$

$$\frac{dp_{11}}{d\tau} = -(1+\rho)\, p_{11} + 2\rho\, p_{12} \tag{4}$$

$$\frac{dp_{12}}{d\tau} = -(2+2\rho)\, p_{12} + 2p_{21} + 3\rho\, p_{13}$$

$$\frac{dp_{13}}{d\tau} = -(3 + 3\rho)\, p_{13} + 4p_{22} + 3\epsilon_2\, p_{31} + 4\rho\, p_{14}$$

$$\frac{dp_{14}}{d\tau} = -(4 + 4\rho)\, p_{14} + 6p_{23} + 6\epsilon_2\, p_{32}$$

$$\frac{dp_{20}}{d\tau} = \rho\, p_{21}$$

$$\frac{dp_{21}}{d\tau} = -(2 + 2\epsilon_2 + \rho)\, p_{21} + 2\rho\, p_{22}$$

$$\frac{dp_{22}}{d\tau} = -(4 + 4\epsilon_2 + 2\rho)\, p_{22} + 3p_{31} + 3\rho\, p_{23}$$

$$\frac{dp_{23}}{d\tau} = -(6 + 6\epsilon_2 + 3\rho)\, p_{23} + 6p_{32}$$

$$\frac{dp_{30}}{d\tau} = \rho\, p_{31}$$

$$\frac{dp_{31}}{d\tau} = -(3+3\epsilon + \rho)\, p_{31} + 2\rho\, p_{32}$$

$$\frac{dp_{32}}{d\tau} = -(6 + 6\epsilon + 2\rho)\, p_{32}$$

We solve this system of first order differential-difference equations numerically by Gill-Method for different values of and ϵ_2.

5.3.3 GRAPHS

(i) In Fig. 5.24, we have plotted $p_{00}(\tau)$, the probability that all susceptibles get infected and all infectives are removed.

(ii) In Fig. 5.25 - 5.29, we have plotted all probabilities of extinction for susceptibles.

(iii) In Fig. 5.30, we have plotted $p_{32}(\tau)$, the probability that no susceptible gets infected and no infective is removed.

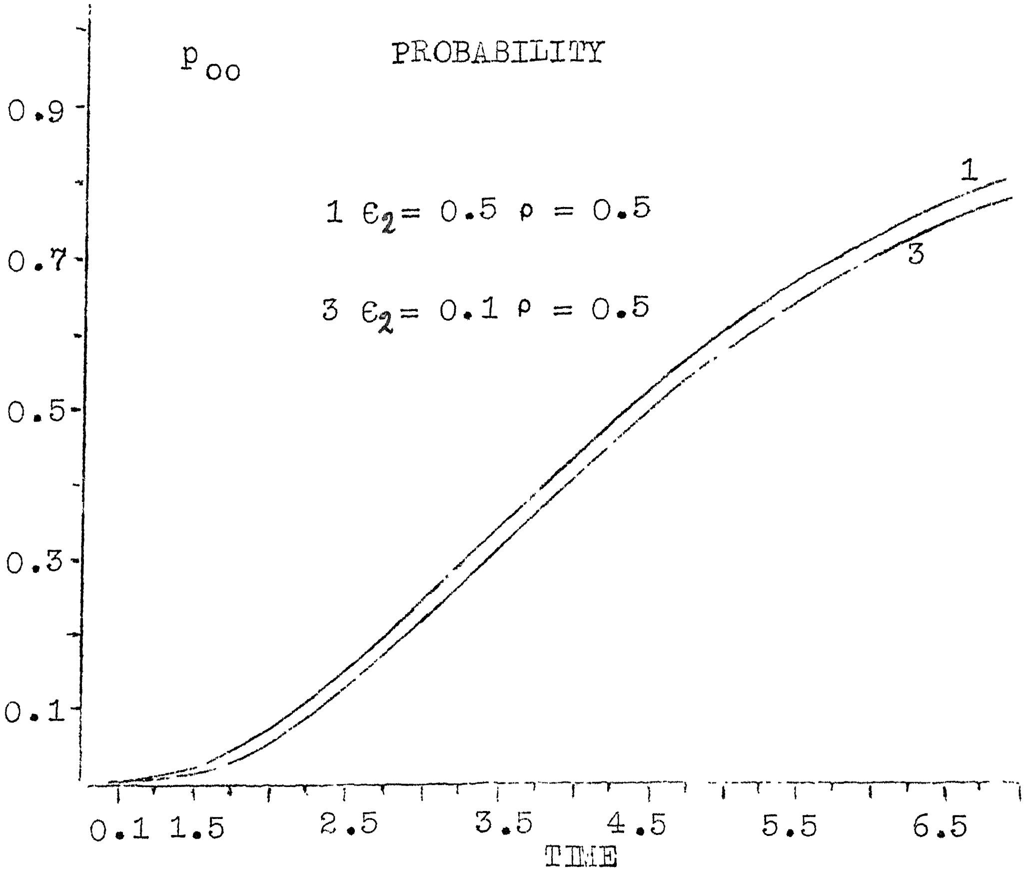


Fig. 5.24

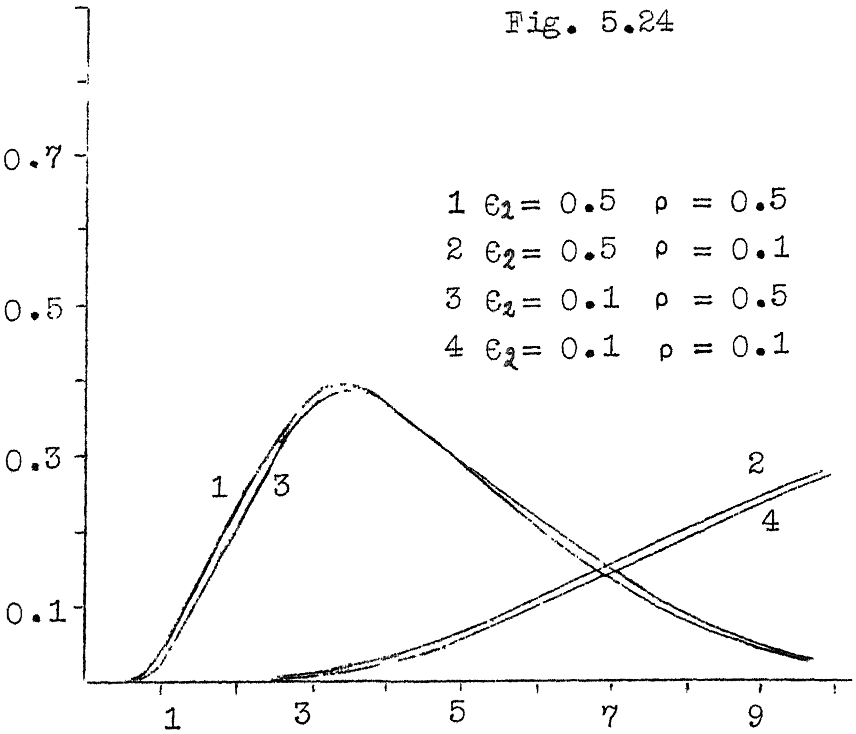


Fig. 5.25

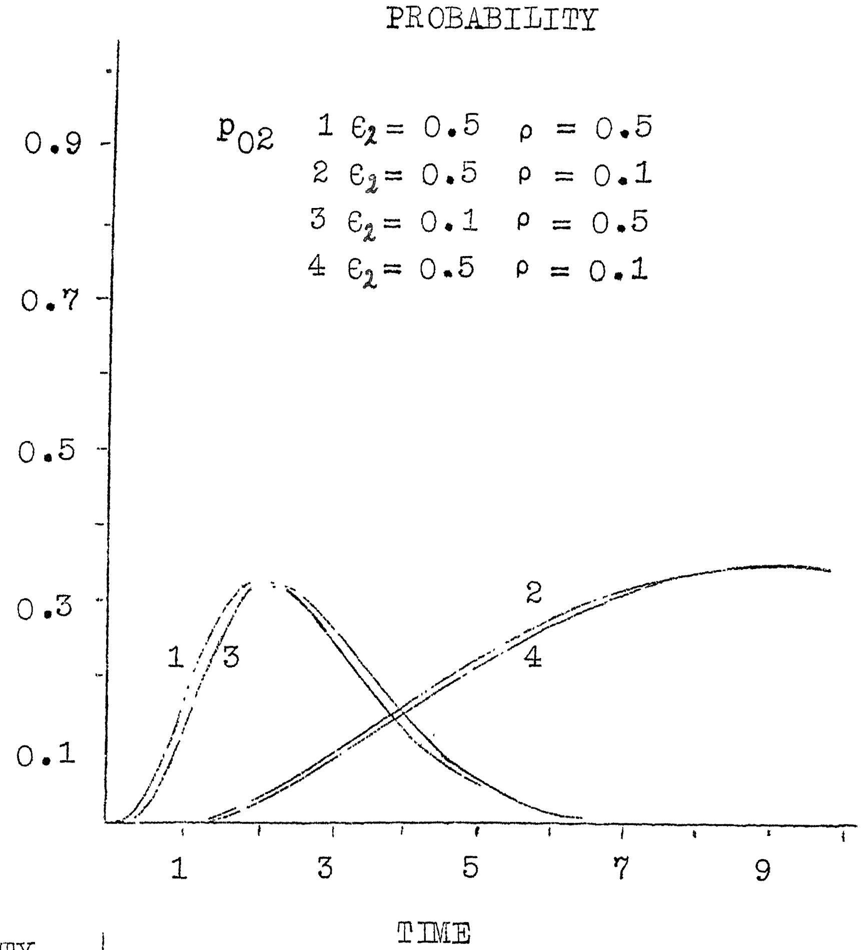


Fig. 5.26

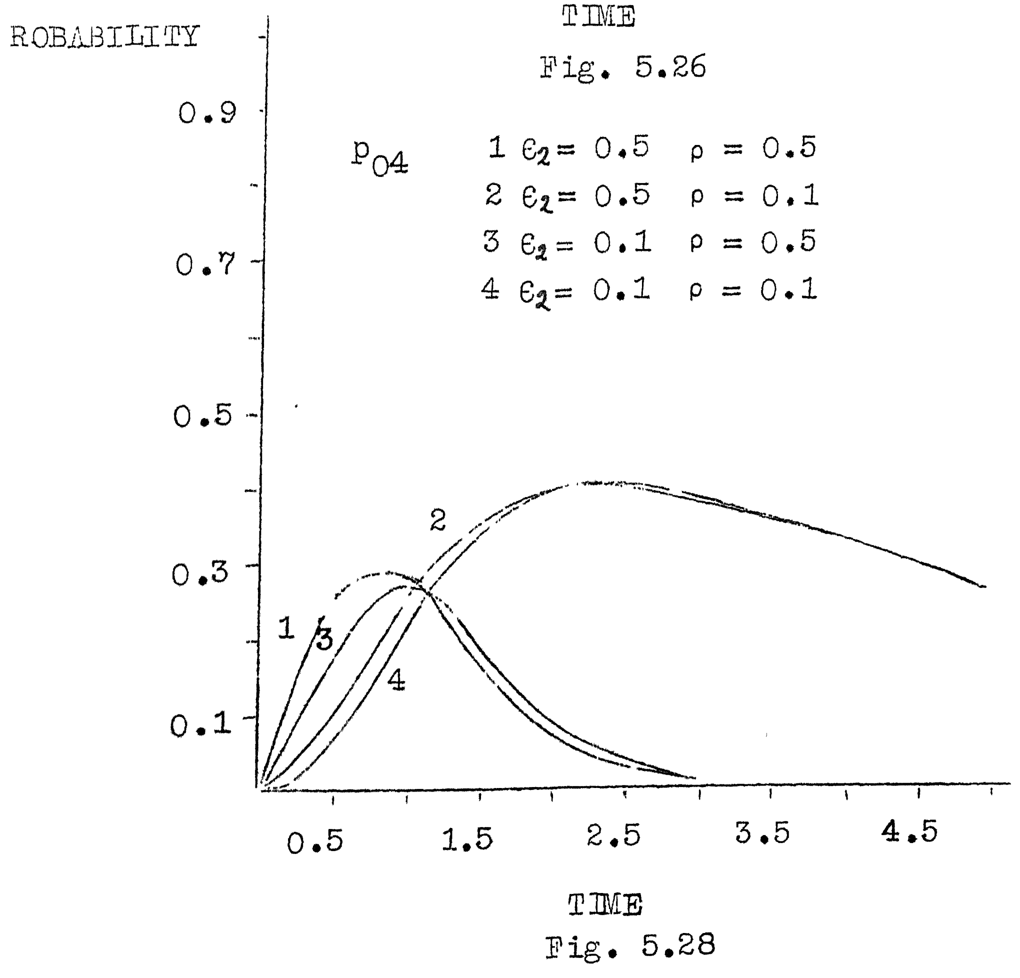


Fig. 5.28

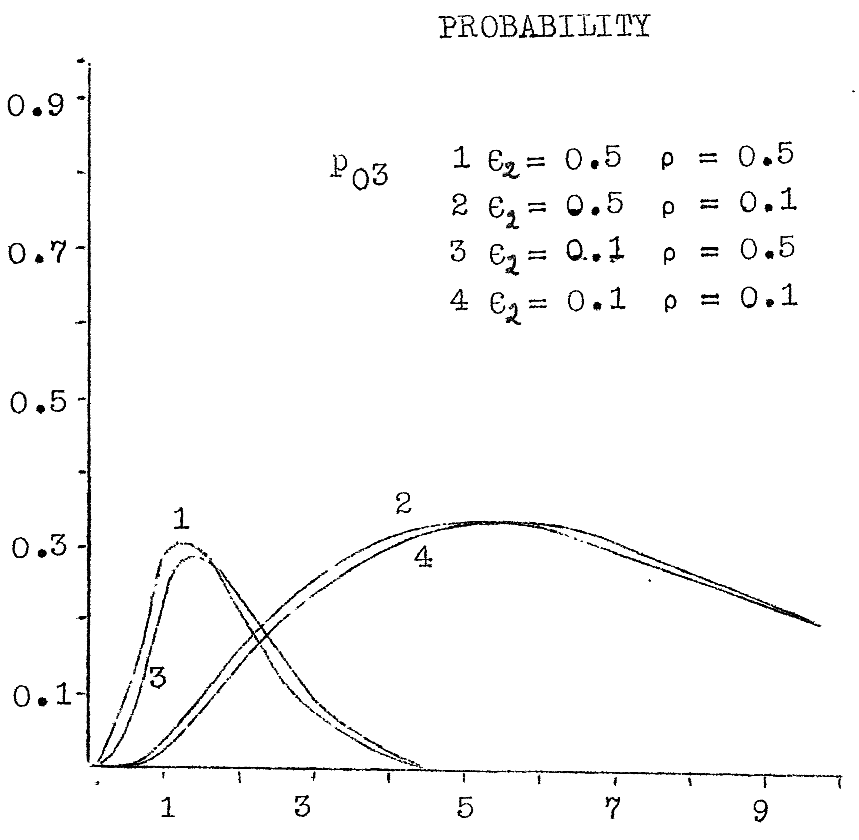


Fig. 5.27

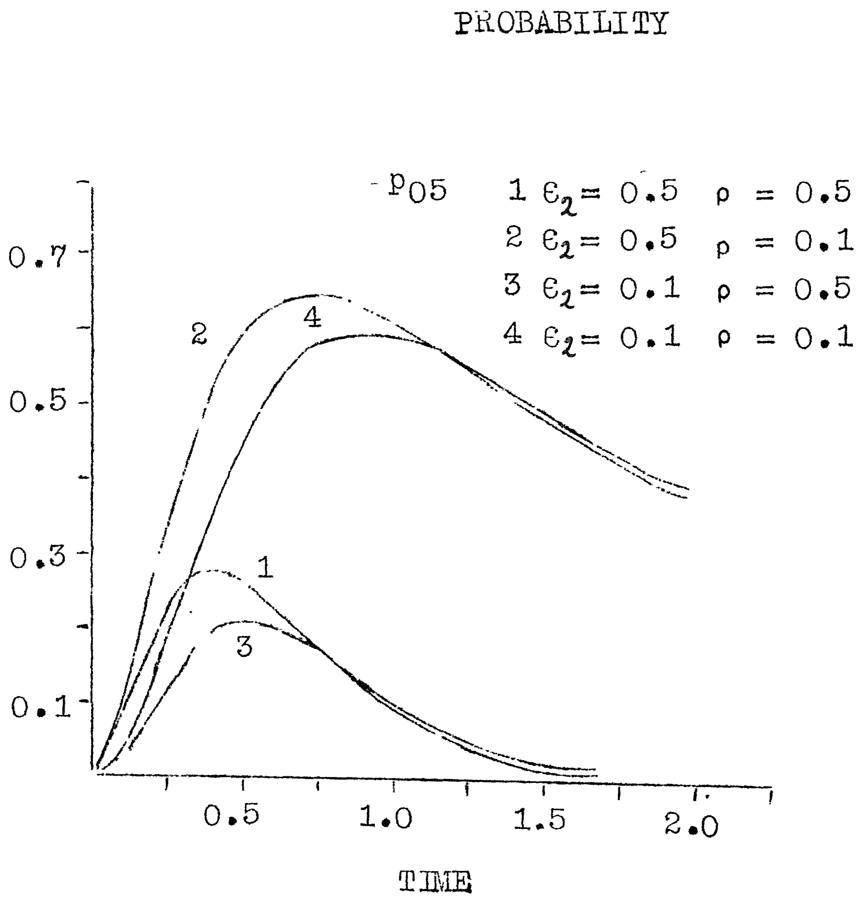


Fig. 5.29

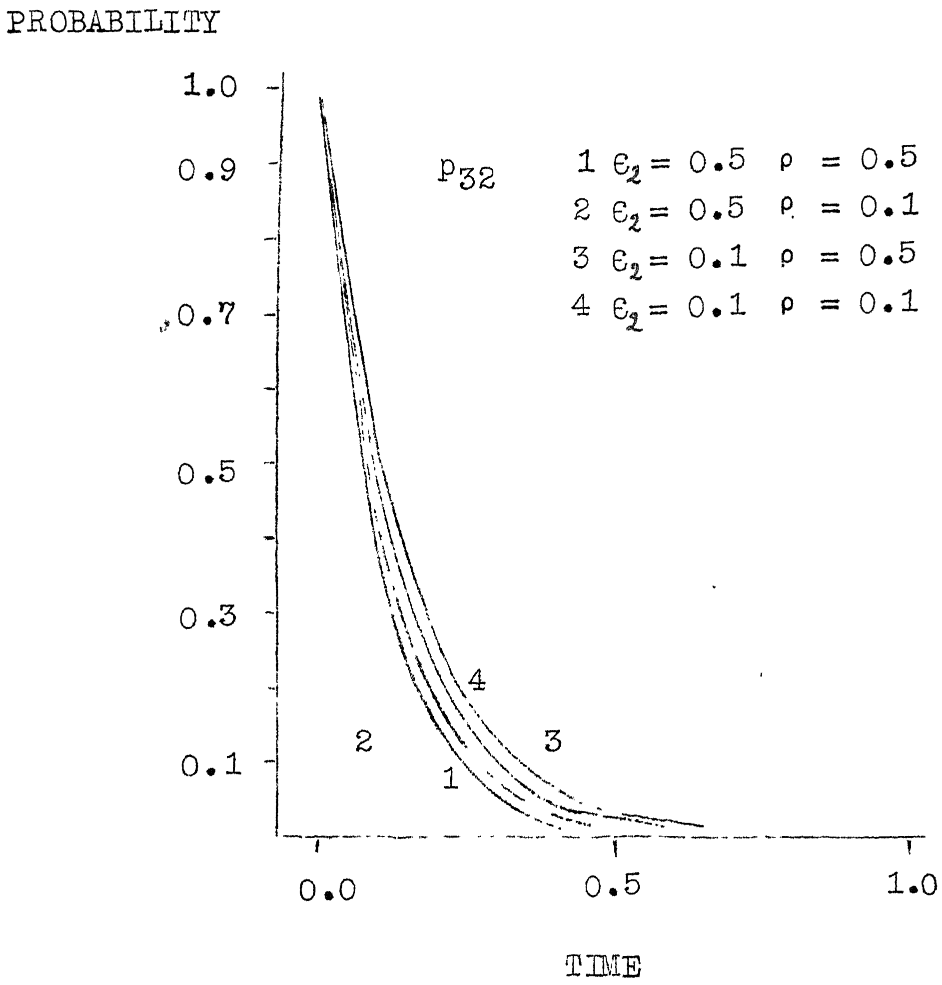

PROBABILITY
1.0
0.9
0.7
0.5
0.3
0.1
p_{32}
1 $\epsilon_2 = 0.5$ $\rho = 0.5$
2 $\epsilon_2 = 0.5$ $\rho = 0.1$
3 $\epsilon_2 = 0.1$ $\rho = 0.5$
4 $\epsilon_2 = 0.1$ $\rho = 0.1$
1
2
3
4
0.0
0.5
1.0
TIME

Fig. 5.30

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